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THE

ELEMENTS OF ALGEBRA

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ELEMENTS OF ALGEBRA

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BY

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PREFACE.

ALGEBRA is a means to be used in other mathematical work; it develops the mathematical language, and is the great mathematical instrument. If the student would become a mathematician, he must understand this language and possess facility in handling the various forms of literal expressions.

Attention is called to the sequence of subjects as herein presented. Involution is introduced as an application of multiplication, evolution as an application of division, and logarithms as an application of exponents. Throughout the book the student is led to see that one subject follows as an application of another subject. The beginner is led to see at the outset that Algebra, like Arithmetic, treats of numbers.

Algebraic terms and definitions are not introduced until the student is required to put them into actual use. Correct processes are clearly set forth by carefully prepared solutions, the study of which leads the pupil to discover that method and theory follow directly from practice, and that methods are merely clear, definite, linguistic descriptions of correct processes.

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The book is sufficiently advanced for the best High Schools and Academies, and covers sufficient ground for admission to any American College.

Great care has been given to the selection and arrangement of numerous examples and problems. These have been, for the most part, tested in the recitation-room, and are not so difficult as to discourage the beginner.

It remains for the author to express his sincere thanks to W. H. Hatch, Superintendent of Schools, Moline, Ill.; to Professor W. C. Boyden, Sub-Master of the Boston Normal School, Boston, Mass.; and to O. S. Cook, connected with the literary department of Messrs. Silver, Burdett & Co., for reading the manuscript and for valuable suggestions.

GEORGE LILLEY.

Washington Agricultural College and
School of Science,

Pullman, Washington, June, 1892.

CONTENTS.

INDEX TO DEFINITIONS.

ELEMENTS OF ALGEBRA.

CHAPTER I.

FIRST PRINCIPLES.

1. In **Algebra** figures and letters are used to represent numbers, instead of figures, as in Arithmetic.

Thus, we may use x to represent the number of dollars in a man's business, the number of cents in the cost of an article, the number of miles from one place to another, the number of persons in our class, etc.

In Algebra, the letter x is reasoned about and operated upon just the same as the numbers which it represents are reasoned about and operated upon in Arithmetic.

2. Symbols of Operation. The signs $+$, $-$, $\times$, and $\div$, are used to denote the algebraic operations addition, subtraction, multiplication, and division, that in Arithmetic can actually be performed. $+$ is read *plus;* $-$ is read *minus;* $\times$ is read *multiplied by;* $\div$ is read *divided by.* A dot or point is sometimes used instead of the sign $\times$. Thus, $a \times b$ and $a \cdot b$ both mean that a is to be multiplied by b. The multiplicand is usually written before the multiplier.

Division in Algebra is more frequently represented by placing the dividend as the numerator, and the divisor as

the denominator of a fraction. Thus, $a \div b$, or $\frac{a}{b}$, means that a is to be divided by b. Read *a divided by b.*

Note. Do not read such expressions as $\frac{m}{n}$, *m over n;* it is meaningless.

3. We must be careful to distinguish between arithmetical and algebraic operations. The former can *actually be performed*, whereas many operations in Algebra can only be indicated.

Thus, suppose a man owes $5 for a vest and $20 for a coat, actual addition gives $25 as his total indebtedness. But if the number of dollars he owes for the vest be represented by m, and the number of dollars that he owes for the coat be represented by n, his entire debt can only be *indicated.* In order to show that the number represented by m is to be added to the number represented by n, we use the sign + written between them; thus, $m + n$.

Exercise I.

Read the following algebraic expressions:

1. $a + 100$; $a + 10 - 2$; $b - 2$; $b - 100 + 8$.

2. $a + b$; $m + n + 6$; $m + s - r$; $a - b + m$.

3. $c + 2 \times 5$; $c - 10 \times 2$; $s - n \times r - 20$.

4. $q + t + 8 \times m$; $c + m \div n - s \cdot q$; $-\frac{c}{a} + c \div a - p + t \cdot x$.

Indicate by means of algebraic expressions the following:

5. The sum of m and n. The difference between m and n. The sum of x, y, and a.

6. The sum of m, n, and r diminished by t. If you had m cents, earned n cents, and are given r cents, and then spend t cents; how many cents will you have left?

7. John has m apples, Henry has n apples, and Charles has b apples; express the number of their apples. How many more have John and Henry than Charles?

8. If you buy goods for a dollars and sell them at a gain of b dollars, express the selling price.

9. I buy goods for m dollars and sell them at a loss of n dollars; express my selling price.

10. Henry had x marbles; he gave John m marbles, and Charles n marbles. How many had he left?

11. I pay n cents for a reader, x cents for a history, y cents for a grammar, 6 cents for car-fare, and have m cents left; express the number of cents that I had at first.

12. A boy earned a dollars, then received m dollars from his father, n dollars from his mother; and spent k dollars of what he had for books, x dollars for a coat, and y dollars for a sled. Express the number of dollars he had left.

4. The **Sign of Multiplication** is generally omitted in Algebra, except between figures. Thus,

$5\,ab$ means $5 \times a \times b$; $p\,r\,s\,t\,u\,z$ means $p \times r \times s \times t \times u \times z$; $2 \cdot 3 \cdot 4 \cdot 5$ means $2 \times 3 \times 4 \times 5$, or 120.

Again, if the *number* of gallons in a cask of cider is represented by a, and the *number* of cents in the cost of one gallon is represented by m, then the *number* of cents in the cost of n casks is represented by $a\,m\,n$.

5. In the expression $5 + 2\,a\,b - a + \frac{m}{n} - \frac{2\,a\,m}{3\,b\,c}$; 5, $2\,a\,b$, a, $\frac{m}{n}$, and $\frac{2\,a\,m}{3\,b\,c}$ are called **Terms.**

Exercise 2.

Read and state the meaning of the following algebraic expressions:

1. $5abx + \frac{mn}{c} - ab$. Result: 5 times a times b times x, plus m times n divided by c, minus a times b; etc.

2. $kl + \frac{mk}{cn} + \frac{a}{b}$; $pqrs + abcd + mnxy - 80$.

3. $amnpqr - cd \times 6 + \frac{3klx}{5wyz}$; $\frac{abcd}{mnop} - 2bd + 11 - r$.

4. $5 \times \frac{mn}{ab} + 12pqrst - 6hk \div a + 100z$.

5. $\frac{3abd - 10mnr + lmnrst}{ad - 1}$.

6. $b + \frac{a-c}{4} + u$; $\frac{2ab}{y} + \frac{2xycd}{u} + \frac{b+z}{x}$; $\frac{c+x-b}{a} + u + \frac{b+y}{l}$.

6. It is customary to write the letters in the order of the alphabet.

In a **product** represented by several letters and numbers, the numbers are written *first*. Thus,

$c \times b \times a \times 5 \times 3$ is written $3 \times 5abc$; both mean $15abc$. Also, $s \times r \times n \times m \times 25$ is written $25mnrs$.

Exercise 3.

Write algebraic expressions for the following:

1. The product of x, y, and z; of m, n, and 5; of 3 and xy; of 5, a, b, and $3 \times mn$.

2. The product of a and b divided by their sum. Their product divided by their difference.

3. The product of m, n, r, and 25 divided by the sum of m and n. The same product divided by the difference of m and n.

4. A travels at the rate of 3 miles an hour; how many hours will it take him to travel 30 miles? How many hours to travel a miles? To travel mn miles? To travel $60\,amn$ miles?

5. A man bought 18 loads of wheat, of m bushels each, at n cents a bushel; how many cents in the entire cost?

6. In example 5, suppose that he sold the wheat at a gain of r cents a bushel; how many cents did he gain? How many cents in the selling price?

7. In example 5, suppose that he sold the wheat at a loss of a cents a bushel; how many cents would he lose? how many cents in the selling price?

8. A man bought a boxes of peaches, each containing b peaches, at c cents a peach; and m baskets of grapes, each containing n pounds, at r cents a pound. How many cents did he pay for both?

9. A man worked n hours a day for m days, at a cents an hour. With the money he bought a coat for x cents; how many cents had he left?

10. One boy sold a apples at c cents each; another sold n peaches at m cents each; a third sold r pears at t cents each. How many cents did they all receive?

11. I buy 5 tons of coal at \$10 per ton, and pay for it in cloth at \$2 per yard; how many yards will it take? I buy a tons of coal at b dollars per ton, and pay for it in cloth at m dollars a yard; how many yards will it take?

12. A man works n weeks at b dollars a week, and his son works m weeks at r dollars a week. With the money they pay for c cords of wood at d dollars a cord; how many dollars have they left?

13. If 5 cords of wood cost \$15, how many dollars will 3 cords cost? If c cords cost \$$m$, how many dollars will n cords cost?

14. A man drove 3 hours at the rate of 10 miles an hour; how many hours will it take him to walk back at the rate of 6 miles an hour? If he drives 3 days n hours each day, at the rate of t miles an hour, and 5 days m hours each day, at the rate of s miles an hour, how many hours will it take him to return over the same distance, at the rate of r miles an hour?

15. If you buy t tons of coal at the rate of \$$d$ for n tons, and sell it at a loss of \$$l$ on each ton, how many dollars will you receive? Suppose you sell at a gain of \$$b$ on each ton, how many dollars will you get for it? Suppose you sell all of it for r dollars, and make a profit, how many dollars profit will you get?

7. Symbols of Relation. The signs $=$, $>$, and $<$, are used for the words, *equals*, *is greater than*, and *is less than*, respectively.

Symbols of Aggregation. The signs ( ), [ ], { }, and $\overline{\quad}$, are used to show that the terms enclosed by them are to be treated as one number. They are called *parenthesis*, *bracket*, *brace*, and *vinculum*, respectively. Thus,

$(2a + b)(3x - y)$, $[2a + b][3x - y]$, $\{2a + b\}\{3x - y\}$, $\overline{2a + b} \times \overline{3x - y}$, each shows that the number obtained by adding the terms $2a$ and b is to be multiplied by the result obtained by subtracting y from $3x$.

Symbols of Abbreviation. The signs (of deduction) $\therefore$, (of reason) $\because$, and (of continuation) ...., are used for the words, *hence* or *therefore*, *since* or *because*, *and so on*, respectively.

8. Since $81 = 9 \times 9$, or written 9^2 for brevity, 81 is called the *second power* of 9. Since $27 = 3 \times 3 \times 3$, or written 3^3 for brevity, 27 is called the *third power* of 3. Similarly a^2, $(m\,n^2)$, $(m + n)^2$, are called *second powers* of a, $m\,n$, and $m + n$; also a^3, $(m\,n)^3$, $(m + n)^3$, are *third powers* of a, $m\,n$, and $m + n$. a^2 means $a \times a$; a^3 means $a \times a \times a$; etc. In general, a^n is called the n*th power* of a, read a nth power.

9. In the expression $a^2 + b^4 c^5 - 3\,x^n$; 2, 4, 5, and n are called **Exponents**. $b^4 c^5$ means $b \times b \times b \times b \times c \times c \times c \times c \times c$; 4 and 5 are used for convenience to show how many times b and c are used as factors.

We must be careful to keep in mind the meaning of each *indicated operation* when reading an algebraic expression. Thus, the expression $5\,x^3\,y^2 - 2\,a^4 b\,(a^7 - b^6)^5 + 3\,a^5\,c^4\,d^m$ means, five times the third power of x times the second power of y, minus two times the fourth power of a times b times the fifth power of the expression in the parenthesis, a seventh power minus b sixth power, plus three times the fifth power of a times the fourth power of c times the mth power of d.

Exercise 4.

Read the following:

1. m^5; $3\,m^5x^2$; $5\,m^2n^3y^5$; $a\,b\,c^4d^5c^3$; $a\,b^2 + b$, $m^2 - n^2$; $10\,a^2b^{20}$.
2. $m^5 n^3 + 5\,a^2 b\,x\,y - 3\,m^{10}\,x^5$; $m^2n^2 - 2\,a\,b\,m\,n + a^2b^2$.
3. $10\,(a\,b)^{10}$; $(m^3n^3)\,(m\,n)^3$; $(a^2 - n)^2$; $(m^2 - 3\,n)^3$.
4. $(m\,n - m^3)^3$; $3\,a^2\,b\,(a - b^2)^3$; $(a^2 + b^2)\,(a^3 - b^3)^2$.

5. $3\,a^n b^n$; $3\,(a\,b)^n$; $a^2\,(b^5 - c^5 - d^7)^{10}\,(m^3\,n^3)\,(m\,n)^5$.

6. $(10\,m + n^4)\,(10\,n^2 - m^5)^4 < 15\,a\,(x - y^2)^2\,(x + y)^3$; $\dfrac{a^2 + b^3 + d^4 + e^7}{5\,(a + b + e)^2} >$ or $< \dfrac{(c^5 + d^3)^4}{c^2 + d^3}$.

7. $\because a + 2x = b + x$, $\therefore x = b - a$; $(a^m - c^n)^2 = (m^2 + n^2)^2$, $\because a^m - c^n = m^2 + n^2$; $m^2 - x^3 = 2\,x^3 - 2\,m^2$, $\therefore x^3 = m^2$; $x + x + x + x + \ldots$ to n terms $= n\,x$; $a \times a \times a \times a \times \ldots$ to n factors $= a^n$; $1 + x + x^2 + x^3 + \ldots = \dfrac{1}{1 - x}$; $a + a\,r + a\,r^2 + a\,r^3 + \ldots$ to n terms $= \dfrac{a\,(r^n - 1)}{r - 1}$.

Write algebraic expressions for the following:

8. The sum of m and n. The double of x. The second power of the sum of a and b. The second power of difference between x and y. Five times the third power of the difference of x and y.

9. The second power of the sum of x second power and y. The second power of the sum of x and y second power. The product of the fourth power of x, the third power of y, and the second power of m. The product of the first power of x and three times the nth power of y. The product of x second power plus y second power, and x second power minus y second power.

10. The product of the sum of x second power and y by $n\,a$. Five n third power minus seven $m\,n$ plus six a second power. m third power minus two times b second power c plus n fourth power is equal to n times y.

11. Seven times m fourth power times n second power minus two times n seventh power times m third power plus three times a third power times b second power plus eight times a second power times b third power plus five

times a fifth power. Since a plus b equals m minus n, therefore the second power of a plus b is equal to the second power of m minus n.

12. Therefore, x is equal to m third power, because x plus three m third power is equal to two x plus two m third power. a plus a plus a, and so on to n minus two terms, equals n minus two times a. The second power of m plus n, divided by m minus n is less or greater than m times a plus b plus c plus d plus e. a less than b is equal to m greater than n.

13. A horse eats a bushels and an ox b bushels of oats in a week; how many bushels will they together eat in n weeks? If a man was a years old 50 years ago, how old will he be x years hence?

10. The **Numerical Value** of an algebraic expression is the number of *positive* or *negative* units it contains, and is found by giving a *particular* value to each letter, and then performing the operations indicated. Thus,

If $a = 3$, $b = 4$, $x = 5$, $y = 6$, find the numerical values of:

$$4\,a^2 b^3\,;\quad \frac{9\,b\,x^3}{25\,a^3\,y^2}.$$

Replacing the letters in each expression by the *particular* values given for them, we have

Process.

$$4\,a^2 b^3 = 4 \times 3^2 \times 4^3 = 4 \times 9 \times 64 = 2304.$$

$$\frac{9\,b\,x^3}{25\,a^3\,y^2} = \frac{9 \times 4 \times 5^3}{25 \times 3^3 \times 6^2} = \frac{9 \times 4 \times 125}{25 \times 27 \times 36} = \frac{20}{27}.$$

11. If one factor of a product is equal to 0, the whole product must be equal to 0, *whatever values the other factors*

may have; and it is also clear that no product can be zero unless one of the factors is zero. Thus, ab is zero if a is zero, or if b is zero; and if ab is zero, either a or b is zero. Again, if $x = 0$, then $a^3b^2xyz^3 = 0$, also $ax(y^2 + 6z + a^3) = 0$, whatever be the values of a, b, y, and z.

Exercise 5.

If $a = 6$, $b = 2$, $c = 1$, $x = 5$, $y = 4$, find the numerical values of the following algebraic expressions:

1. $3c^2$; $7y^3$; $5ab$; $9xy$; $8b^3$; $3x^5$; x^8; $7y^4$; ac^{10}; $3cb^4$.
2. $9b^4$; $2ax$; y^3; $10x^2$; $\frac{1}{4}y^2$; $5by$; $\frac{1}{8}a^3$; $\frac{3}{10}abcxy$.
3. $\frac{5}{12}c^3$; $\frac{1}{10}ax$; $7c^5$; $\frac{3}{5}x^3$; a^4x^2; $8a^2$, $2y^4$; $\frac{5}{6}acxy$.

If $m = 2$, $n = 3$, $p = 1$, $q = 0$, $r = 4$, $s = 6$, find the values of:

4. $\dfrac{9m^2r}{8n}$; $5m^2n$; $\dfrac{4ps^2}{9m^3}$; $\dfrac{8mn^3}{9r^2}$; $\dfrac{6m^3p}{n^2}$; 3^m2^n; $\dfrac{2m^2q}{7s}$.
5. $\dfrac{8n^4}{9m^3}$; $\frac{5}{6}nm^6$; 2^6m^5; p^3n^4; $\dfrac{5m^6r^n}{64s^m}$; $\dfrac{27m^r}{32}$; $\dfrac{64}{r^s}$; $\dfrac{2^r}{3^n}$.

EXAMPLE 6. Find the value of $5b^3 + \frac{3}{10}xy - 5a^2 - \frac{3}{4}a^2b^3$, when $a = 2$, $b = 3$, $x = 5$, and $y = 10$.

Replacing the letters in the expression by the *particular* values given for them, we have

Process.

$$\begin{aligned}5b^3 + \tfrac{3}{10}xy - 5a^2 - \tfrac{3}{4}a^2b^3 &= 5 \times 3^3 + \tfrac{3}{10} \times 5 \times 10 - 5 \times 2^2 - \tfrac{3}{4} \times 2^2 \times 3^3\\ &= 5 \times 27 + 3 \times 5 \qquad - 5 \times 4 - 3 \times 27\\ &= 49.\end{aligned}$$

EXAMPLE 7. Find the value of $mny^5 + m^5nxyt + m^2n^3r^ss - \dfrac{m^2n^3}{4y^2}$, when $m = 5$, $n = 2$, $r = 3$, $s = 4$, $x = 0$, and $y = 1$.

Process.

$$mny^5+m^5nxyt+m^2n^3r^s s-\frac{m^2n^3}{4\,y^2}=5\times 2\times 1^5+0+5^2\times 2^3\times 3^4\times 4-\frac{5^2\times 2^3}{4\times 1^2}$$
$$=5\times 2\times 1+25\times 8\times 81\times 4-\frac{25\times 8}{4\times 1}$$
$$=10+64800-50$$
$$=64760.$$

If $a=1$, $b=2$, $c=3$, and $d=0$, find the numerical values of the following algebraic expressions:

8. $10\,a-4\,b+6\,c+5\,d;\ a\,b+b\,c+a\,c-d\,a.$

9. $6\,a\,b-3\,c\,d+10\,a\,d-2\,b\,c+2\,b\,d;\ 2\,b\,c+10\,c\,d.$

10. $a^2+b^2+c^2-d^2;\ a\,b\,c+10\,b\,c\,d+5\,a\,c\,d+3\,a\,b\,d.$

11. $a^4+b^3+c^2-d;\ +5\,b-8\,c+a\,d;\ -40\,a\,d+a\,b.$

12. $5\,a+3\,c-6\,b+6\,d;\ 3\,b\,c\,d+2\,a\,c\,d-10\,a\,b\,d.$

13. $5\,b\,c^3+a^3+b^3-1.25\,a\,b^3\,c;\ 15\,a^2+\frac{11}{9}\,c^4+10\,a\,b.$

14. $c^3-8\,a\,d^6\,x-5\,a^{10}+\frac{1}{3}\,b^3\,d\,y;\ \dfrac{.6\,a^2\,b^4}{c^3}-\dfrac{.1\,a^3}{a^{10}\,b^2\,c}.$

15. $125\,a\,b\,c\,d^4\,m+\dfrac{a^3}{8\,b}-\dfrac{a\,d\,n}{x};\ a^3+b^3+c^3+d^3\,x.$

16. $\frac{2}{9}\,a\,b^3+\frac{3}{11}\,a\,b^2\,c-\dfrac{a^b}{b^c};\ 3\,a^2\,b^3\,d^6\,p+\dfrac{c^b}{b}-\dfrac{2\,a^c}{a^b}.$

17. $\frac{3}{8}\,a\,c^b\,d^c-\frac{5}{6}\,a^c\,b^3\,d^n+\dfrac{8}{2^c};\ \frac{5}{27}\,a\,b-\dfrac{3}{c^2}+\dfrac{5\,a}{a\,b\,c}.$

18. $\frac{5}{6}\,a\,b\,c+\frac{3}{4}\,c^3\,d\,q^n+\dfrac{8\,d\,r^m}{a};\ \dfrac{8^c}{3^b}+\dfrac{2^a}{9^a}+\dfrac{a}{1^c}+\frac{3}{4}\,a^3\,c^3\,d^5\,x.$

19. $\frac{1}{3}\,a\,d\,y+\frac{1}{3}\,a-\frac{8}{9}\,a^2\,d^3\,x\,y;\ \dfrac{1}{3^a}\,a\,d\,s+\dfrac{9\,a\,b}{8}-1.$

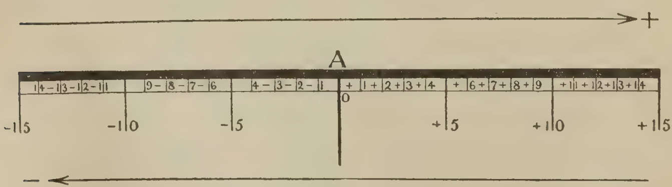
12. Negative Numbers. If a person owes a debt of ten dollars, and has but six dollars in money, he can pay the debt only in part.

For his six dollars in money will cancel only six dollars of his debt, and leave him still owing four dollars; we may consider him as being worth four dollars less than nothing. The total number of dollars that he is worth may be represented by -4, because it will take four dollars in addition to the six dollars to pay the debt. If a person *gains* eight dollars and *loses* eleven dollars, the number of dollars in his net loss may be represented by -3, because it will take three dollars in addition to the gain to balance the loss. Similarly, if he *gains* 100 dollars and *loses* 120 dollars, the number of dollars in his net loss may be represented by -20. To enable us to represent these numbers, it is necessary to assume a new series of numbers, beginning at zero and descending in value from zero by the repetitions of the unit, precisely as the natural series ascends from zero. To each of these numbers the sign $-$ is prefixed. The negative series of numbers is written thus :

$$\ldots\ldots -10,\ -9,\ -8,\ -7,\ -6,\ -5,\ -4,\ -3,\ -2,\ -1,\ 0.$$

For convenience the algebraic series of numbers is represented as follows :

Scale of Numbers. We may conceive algebraic numbers as measuring distances from a **fixed point** on a straight line, extending indefinitely in both directions, the distances to the right being *positive*, and the distances to the left *negative*. From *any point* on the line, measuring *toward the right is positive* and *toward the left negative.*



In the above illustration consider A the zero or starting-point on the scale of numbers, and the distance between any two consecutive numbers *one unit.* The distances to the right and left of A are posi-

tive (+) and negative (−), respectively, as indicated by the directions of the arrows.

To add $+9$ to $+4$ (read 9 *and* 4 *in the positive series*), we start at 4 in the *positive series*, count *nine* units in the *positive direction*, and arrive at 13 in the positive series. That is, $+4+(+9)=13$.

To add $+9$ to -4 (read 9 *in the positive series* and 4 *in the negative series*), we begin at 4 in the *negative series*, count *nine* units in the *positive direction*, and arrive at 5 in the positive series. That is, $-4+(+9)=+5$.

To add -9 to $+4$, we start at 4 in the *positive series*, count *nine* units in the *negative direction*, and arrive at 5 in the negative series. That is, $+4+(-9)=-5$.

To add -9 to -4, we start at 4 in the *negative series*, count *nine* units toward the *left*, and arrive at 13 in the negative series. That is, $-4+(-9)=-13$.

To subtract $+9$ from $+4$, we start at 4 in the *positive series*, count *nine* units in the *negative direction*, and arrive at 5 in the negative series. That is, $+4-(+9)=-5$.

To subtract $+9$ from -4, we begin at 4 in the *negative series*, count *nine* units in the *negative direction*, and arrive at 13 in the negative series. That is $-4-(+9)=-13$.

To subtract -9 from $+4$, we begin at 4 in the *positive series*, count *nine* units in the *positive direction*, and arrive at 13 in the positive series. That is, $+4-(-9)=+13$.

To subtract -9 from -4, we start at 4 in the *negative series*, count *nine* units in the *positive direction*, and arrive at 5 in the positive series. That is, $-4-(-9)=+5$.

13. The sign + is often omitted before a number in the positive series. Thus, the numbers 3, 5, and 6, taken alone, mean the same as $(+3)$, $(+5)$, and $(+6)$, showing that the numbers are in the *positive series*

The sign − must always be written when a number is in the *negative series*. Thus, the numbers 3, 5, and 6, taken in the *negative series*, are written (-3), (-5), and (-6).

The **Algebraic Signs** + and − mark the direction that the numbers following them are to take. These signs are

used to indicate **opposition** (opposite direction), also **operation**. The former is called the **positive**, and the latter the **negative** sign.

An **Algebraic Number** is one which is represented by an algebraic term *with its sign of direction*. Thus, $+3$, -3, $-a$, and $+5a$ are algebraic numbers.

Absolute Value shows what place a number has in the positive or negative series. Thus, $+3$ and -3 have the same absolute value; that is, three *units*.

Absolute Numbers are those not affected by the signs $+$ or $-$.

EXAMPLE. The meaning of an algebraic expression, as

$$3x^2 + (-2ab) - [c - (-y)],$$

is explained thus:

To $3x^2$ units in the positive series *add* $2ab$ units in the negative series, and from their sum *subtract* the expression in brackets, c in the positive series *minus* y in the negative series. The signs written before the terms $(-2ab)$, $(-y)$, and before the bracket, indicate *operation*. The sign written before $2ab$ and y, also the sign understood before $3x^2$ and c, indicate *opposition*.

Exercise 6.

1. Over how many units and in what series of numbers would a point move in passing from $+3$ to -8? -10 to $+1$? $+5$ to $+15$? -12 to -1? -1 to -12? 15 to 5? 9 to 9? -5 to -5?

2. Which is the greater, 0 or -6? 3 or -3? -5 or -3? $+10$ or -1? $+50$ or -50, and how many units?

3. How many units is $+6$ greater than 0, $+3$, -3, -6, and -5? How many units is -5 less than 5? How many units is a less than b?

4. If a point start at $(+3)$ and move three units to the right, then five units to the left, where is it? Express its distance from 0.

5. Suppose a point start at $(+2)$, and move six units to the right, then eleven units to the left, where is it?

6. Where is the point which, starting at (-5), moved (-3), then $(+8)$? Express its distance from the starting-point.

7. Suppose a point starting at $+3$, move $+2$, then -7, then $+5$, then -6, then $+10$, then -11, where would it be? Express its distance from $+3$.

Explain the meaning of:

8. $2[3b + (-5a)] - 5[(-a) + (+b)]$.

9. $(+8x) + [+3x - (+12y) + (-x) - (8y)]$.

Also the meaning of the signs $+$ (as used or understood) and $-$.

Explain the meaning of:

10. $6a^5b^2 + (+a^3b^6) + (+a^2b^2c^2) + (-a^3b^2) + (+a^3b^6) + 20a^2b^2c^2 + (-a^2b^3) + (-a^3b^2)$.

11. $a^2k^2 + (-y^3x^{10}) + (+a^6x^3) + (-yx) + m^3n^5 - (+a^{10}m^5)$.

12. $+(+a^5) - (+b^3) - (+a^5) - (+b^3)$.

13. $(x+y)^2 + (a+x)^5 - (x+y)^2 - (a+x)^5$.

Find their numerical values when $a = b = c = m = n = k = y = 1$.

Read the following expressions, and find their numerical values when $a = 0$, $b = 1$, $c = 2$, $d = 3$, $e = 4$, $n = 5$, and $m = 6$:

14. $c^5 \div (+c) - (+n^3)(+c) \div (+d^5) + a^2(+b^2)(+c^3)$.

15. $3[e + (+n)] - 5(+c) \div (a + b) + 2(+a) \div b$.

16. $(+c)[a + (+n) + (+d) - (+e)] - (+m) \div (+d)$.

17. $[(+m) - (+d) + (+m) - (+e)(+c)] \div [(+m)(+b)]$.

18. $d^3 \div (+d^c) + 2(+b^c) \div (+c^b) - (+c^d) \div d - e^m \div (+4^c)$.

If $a = 5$, $b = 4$, $c = 3$, $d = 2$, and $e = 1$, find the numerical values of the following expressions:

19. $(+a^2) + (+b^3) - (+c^2) \div (+e^5)$; $a^4c - a^4d^2 + d^5 \div (a^2b^2)$.

20. $c(a-b)$; $e^3 + \left(\frac{2}{d^2} + \frac{d^2}{b}\right)c^2 + \left(\frac{1}{b} - \frac{1}{4d}\right)c + \frac{3}{c^2}$; $\frac{3}{3e+3c} + \frac{7}{6}$.

If $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, find the values of:

21. $\frac{2e+2}{e-3} + \frac{3e-9}{e-2} + \frac{c^2-1}{e+3}$; $\frac{d^c}{b^e}$; $\frac{8a^2+3b^2}{a^2+b^2} + \frac{4c^2+6b^2}{c^2-b^2} - \frac{c^2+d^2}{e^2}$.

22. $\frac{e^c+b^a}{c^b-b^c}$; $\frac{a^2+b^2}{e} + \frac{c^2+e^2}{b} + \frac{e^2-d^2}{c}$; $\frac{b^c+d^c}{b^2+d^2-bd}$; $\frac{c^c-d^c}{c^2+cd+d^2}$.

23. $\frac{a^4+4a^3b+6a^2b^2+4ab^3+b^4}{a^3+3a^2b+3ab^2+b^3}$; $\frac{28}{a^2+b^2+c^2} + \frac{12}{d^2-c^2-b^2}$.

24. $ab - 15b \div 5$; $\frac{a^2+2ab+b^2}{a+b} + \frac{c^2+2cd+d^2}{c+d} - \frac{b^2+2bc+c^2}{b+c}$.

25. $\frac{a^4-4a^3c+6a^2c^2-4ac^3+c^4}{b^4-4b^3c+6b^2c^2-4bc^3+c^4}$; $12e - 4a \div (2a \times b) - 2b$.

26. $[(12e - 4a) \div 2a] \times b$; $a^5 - (a^2b^4e^3) \div (c^3 \div a^4)$.

27. $\dfrac{4a+3b}{b+c}-\dfrac{4c+3d}{b+d}+\dfrac{5d+4e}{a+d+e}$; $(a+b)(b+c)$.

28. $(b+c)(c+d)+(c+d)(d+e)+\dfrac{4}{a^2+e^2-c^2-d^2}$.

29. $12e-(4a\div 2a)\times b-2b$; $3e\left[\left(\dfrac{2a^2}{c+d}-\dfrac{b+12e^2}{d}\right)\div d\right]$.

30. $6b\div(a-c)-3d+abcd\div 24a$; $c+5d\div 5 + a\times e$.

Express the following statements in algebraic symbols:

31. To the double of a add b.

32. To five times x add b diminished by one.

33. Increase b by the sum of a and x divided by y.

34. Write x, a times.
What is the sum of $x+x+x+\dots$ written a times?

35. Write three consecutive numbers of which n is the least.

36. Write five consecutive numbers of which m is the greatest.

37. Write m, a minus 1 times; also m plus n times.

38. Write seven consecutive numbers of which x is the middle one.

39. Write a, xth power, minus y, nth power.

40. To the double of x, increased by a divided by b, add the product of a, b, and c.

41. To the product of a and b add the quotient of x divided by a, and divide their sum by y diminished by c.

42. Write a exponent n plus the quotient of x divided by y, minus b times the quotient of b divided by the expression, a exponent c plus b exponent m, is greater than b minus x.

43. Write x fifth power minus b sixth power plus y to the mth power, divided by z to the nth power, is less than q tenth power.

44. Write a to the nth power divided by b exponent m, minus x exponent n, equals a minus b, divided by the sum of a second power and b third power.

45. Write c fourth power divided by a second power, minus the product of x and y, plus x .... written n times, equals a exponent m.

46. x exponent m, plus the fraction, a fifth power minus three times a second power b third power, divided by x minus y, equals x minus y, added to the sum of $4a$ and b minus m, plus 1 divided by x to the nth power.

47. Five times the third power of a, diminished by three times the third power of a times the third power of b, and increased by two times the second power of b.

48. Three times x exponent 2, minus twice the product of x exponent 3 and y, plus the third power of a.

49. Six times the third power of x multiplied by the second power of y, minus a exponent 2 times the fourth power of b.

50. a times the second power of n, divided by x minus y, increased by six a times the expression x plus y minus z.

CHAPTER II.

ALGEBRAIC ADDITION.

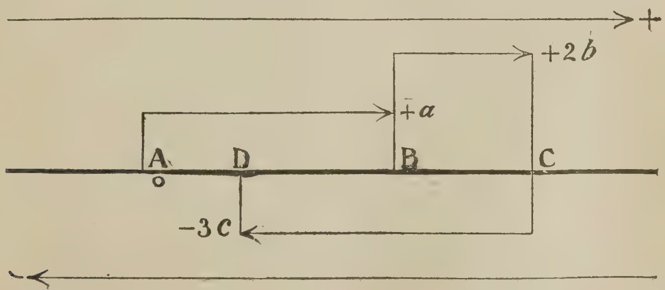
14. In Art. 12 it was shown that to add a positive number means to count so many units in the *positive* direction, and to add a negative number means to count so many units in the *negative* direction.

In **Algebraic Addition** of several numbers, we count from the place in the series occupied by any one of the numbers, as many units as are equal to the *absolute value* of the numbers to be added and in the *direction* indicated by their *signs*. Thus,

Example 1. Find the sum of $3a$ and $-9a$

Solution. $3a$ signifies a taken 3 times in the positive series, and $-9a$ signifies a taken 9 times in the negative series. We count from $+3a$, $9a$ units in the *negative* direction, and a is taken in all 6 times in the negative series, or $-6a$. That is, $3a+(-9a) = -6a$. Similarly $(+9a)+(-3a) = +6a$.

Example 2. Find the sum of a, $2b$, and $(-3c)$.



Explanation. Suppose these algebraic numbers to be accurately measured as represented on the line of numbers $A\,C$. Start at B, then count $2\,b$ *units* in the *positive* direction and arrive at C. Now count $3\,c$ *units* in the *negative* direction, and arrive at D in the *positive* series.

Thus, $(+a) + (+2\,b)$, or $a + 2\,b = A\,C$;
$a + 2\,b + (-3\,c)$, or $a + 2\,b - 3\,c = A\,D$.

The sum of the algebraic numbers is equal in *absolute* value to $A\,D$ in the positive series. That is,

$(+a) + (+2\,b) + (-3\,c) = a + 2\,b - 3\,c$. Hence,

The sum of several algebraic numbers is expressed by connecting them with their proper signs.

Notes 1. The sum of several algebraic numbers is the *excess* of the numbers in the positive series over those in the negative series, or the *excess* of the numbers in the negative series over those in the positive series, according as the one or the other has the greater *absolute* value. Thus, in Example 1 the algebraic sums are $-6\,a$ and $+6\,a$. In Example 2 the algebraic sum is $A\,D$ in the *positive* series.

2. The *sum* of algebraic numbers is the *simplest expression of their aggregate values.*

3. Algebraic addition is *not always augmentation* as in arithmetic. Thus, $(+7) + (-5) = 2$; also $(+8) + (-12) = -4$.

15. A **Coefficient** of a term is a *factor* showing how many times the remainder of the term is taken. Thus,

In the term $5\,a\,b\,m$, 5 is the coefficient of $a\,b\,m$, and shows that $a\,b\,m$ is taken 5 times; $5\,a$ is the coefficient of $b\,m$; $5\,a\,b$ is the coefficient of m. In the term $4\,m\,(a\,b - 2\,a)$, 4 is the coefficient of $m\,(a\,b - 2\,a)$; $4\,m$ is the coefficient of $(a\,b - 2\,a)$.

Note. A coefficient may be **numerical** or **literal**. When **no numerical** coefficient is expressed, 1 is always understood to be the coefficient; as, x; $x\,y^5$.

Like Terms are those having the same letters affected with the same exponents. Thus,

$2m^2nx^3y^5$, $m^2nx^3y^5$, and $-10m^2nx^3y^5$ are like terms, as are also $5x^{10}y^6z^3$ and $-3x^{10}y^6z^3$; but $3x^9yz^3$ and $5x^9y^2z^3$ are *unlike* terms. Like terms are said to be **similar.**

A **Monomial** or **Simple Expression** consists of one term: as, x; $10abc$; $-5a^5x^2$.

16. To Add Similar Monomials.

I. When all the Terms are Positive or Negative. *Add the numerical coefficients; to the sum, annex the common symbols, and prefix the common sign.*

II. When Some of the Terms are Positive and Some are Negative. *Add separately the numerical coefficients of all the positive terms and the numerical coefficients of all the negative terms; to the difference of these two results, annex the common symbols, and prefix the sign of the greater sum.*

EXAMPLE 1. Find the sum of $10xy^5$, $-3xy^5$, $4xy^5$, $-11xy^5$, and $-17xy^5$.

Explanation. For convenience write the terms as shown in the margin. The sum of the coefficients of the positive terms is 14, and the sum of the coefficients of the negative terms is 31. The difference of these is 17, and the sign of the greater sum is negative. Hence, the required sum is $-17xy^5$.

Process.

$$\begin{array}{r} +10xy^5 \\ +\ 4xy^5 \\ -\ 3xy^5 \\ -11xy^5 \\ -17xy^5 \\ \hline -17xy^5 \end{array}$$

EXAMPLE 2. Find the sum of $(x+y)$, $1.1\,(x+y)$, $-2.9\,(x+y)$, $.29\,(x+y)$, $-\frac{1}{4}(x+y)$, and $1.26\,(x+y)$.

Explanation. $(x+y)$, enclosed in parentheses, is treated as a simple symbol. The coefficients of $(x+y)$ are 1, 1.1, 2.9, .29, $\frac{1}{4}$, and 1.26. The sum of the coefficients of the positive terms is 3.65, and the sum of the coefficients of the negative terms is 3.15. The difference of these is .5, and the sign of the greater sum is positive. Etc.

Process.

$$\begin{array}{r} + \quad (x+y) \\ 1.1\ (x+y) \\ +\ .29\,(x+y) \\ +1.26\,(x+y) \\ -2.9\ (x+y) \\ -\quad \frac{1}{4}(x+y) \\ \hline +.5\,(x+y) \end{array}$$

Exercise 7.

Find the sum of:

1. $(+2a)$, $(+a)$, $(+4a)$, $(+3a)$, $(+5a)$, and $(+.1a)$.

2. $(+5ax)$, $(+2ax)$, $(+6ax)$, $(+ax)$, and $(+ax)$.

3. $(+6c)$, $(+8c)$, $(+2c)$, $(+15c)$, $(+9c)$, and $(+c)$.

4. $(-6abc)$, $(+4abc)$, $(+abc)$, $(-2abc)$, and $(+5abc)$.

5. $(-\frac{5}{3}x^2)$, $(-\frac{3}{4}x^2)$, $(-\frac{4}{3}x^2)$, $(-\frac{1}{4}x^2)$, and $(-x^2)$.

6. $(+\frac{2}{3}x)$, $(-\frac{3}{4}x)$, $(+\frac{5}{6}x)$, $(-2x)$, $(+\frac{4}{6}x)$, and $(+x)$.

7. $(+3a^3)$, $(-7a^3)$, $(-8a^3)$, $(+2a^3)$, and $(-11a^3)$.

8. $(+4a^2b^2)$, $(-a^2b^2)$, $(-7a^2b^2)$, and $(+.5a^2b^2)$.

9. $(+7abcd)$, $(+2abcd)$, $(-1.1abcd)$, and $(-4.1abcd)$.

10. $+(b+c)$, $-.01(b+c)$, $+.7(b+c)$, $-10(b+c)$, and $+\frac{3}{5}(b+c)$.

11. $+10(x-y)^3$, $-(x-y)^3$, $+.01(x-y)^3$, $-.2(x-y)^3$, and $-3(x-y)^3$.

12. $+\frac{1}{3}\left(\frac{a}{b}\right)^2$, $-\frac{11}{8}\left(\frac{a}{b}\right)^2$, $+\frac{1}{4}\left(\frac{a}{b}\right)^2$, and $-2.5\left(\frac{a}{b}\right)^2$.

17. If the monomials are not all like, combine the like terms, and write the others, each preceded by its proper sign (Art. 14).

Example 1. Find the sum of $(+7x)$, $(+3by^2)$, $(-2x)$, $(-5by^2)$, $(+4x)$, $(-8by^2)$, $(+9x)$, $(+by^2)$, $(+11x)$, and $(-by^2)$.

Explanation. For convenience, write the expressions so that like terms shall stand in the same column, as in the margin. The sum of the terms containing x is $+29x$, and the sum of those containing by^2 is $-10by^2$. Hence, the result is $+29x - 10by^2$.

Process.

$$\begin{array}{rr}
+\ 7x & +\ 3by^2 \\
-\ 2x & -\ 5by^2 \\
+\ 4x & -\ 8by^2 \\
+\ 9x & +\ by^2 \\
+\ 11x & -\ by^2 \\
\hline
+\ 29x & -\ 10by^2
\end{array}$$

EXAMPLE 2. Find the sum of $+.05(a+b)$, $-.01(m+n)$, $+7(a+b)$, $-3(m+n)$, $-11(a+b)$, and $+10(m+n)$.

Explanation. $(a+b)$ and $(m+n)$, enclosed in parentheses, are treated as simple symbols. The sum of the like terms containing $(a+b)$ is $-3.95(a+b)$. The sum of the like terms containing $(m+n)$ is $+6.99(m+n)$.

Process.

$$\begin{array}{rr}
+\ .05(a+b) & -\ .01(m+n) \\
+\ 7(a+b) & -\ 3(m+n) \\
-\ 11(a+b) & +\ 10(m+n) \\
\hline
-3.95(a+b) & +6.99(m+n)
\end{array}$$

Exercise 8.

Find the sum of:

1. $(+.3x)$, $(+.5y)$, $(+.01x)$, $(+3y)$, and $(-7x)$.

2. $(+\frac{1}{2}a)$, $(-\frac{1}{5}ab)$, $(+\frac{5}{6}a)$, $(+\frac{2}{11}ab)$, and $(-\frac{7}{8}a)$.

3. $(+5c^2x^2)$, $(-2a^8x)$, $(-2c^2x^2)$, $(+10a^8x)$, $(+8c^2x^2)$, $(-4a^8x)$, $(-4c^2x^2)$, and $(+4a^8x)$.

4. $(+\frac{5}{3}x^2)$, $(-\frac{1}{3}ab)$, $(+\frac{11}{6}x^2)$, $(+\frac{5}{12}ab)$, $(+\frac{7}{10}x^2)$, $(+\frac{1}{2}x^2)$, $(+\frac{1}{6}ab)$, $(-\frac{3}{4}x^2)$, $(-\frac{1}{4}ab)$, and $(+\frac{3}{5}ab)$.

5. $7a$, $-3(x-y)$, $8a$, $.3(x-y)$, $.03(x-y)$, and $-.1a$.

6. $\left(\frac{x}{a}\right)$, $\left(\frac{x}{n}\right)$, $.2\left(\frac{x}{a}\right)$, $-.2\left(\frac{x}{n}\right)$, $5.5\left(\frac{x}{a}\right)$, and $-3.1\left(\frac{x}{n}\right)$.

18. A **Polynomial** or **Compound Expression** consists of *two* or *more* terms.

EXAMPLE. Find the sum of $8ax - .1y + 5$, $.7ax + y - am - 9$, and $-.3ax - 1.02y + 5p - .3$.

Process.
$$\begin{array}{rrrrr} 8ax - & .1y & & +5 & \\ .7ax + & y & -am & -9 & \\ -.3ax - & 1.02y & & -.3 & +5p \\ \hline 8.4ax - & .12y & -am & -4.3 & +5p \end{array}$$
Hence, in general,

To Add Polynomials. *Write the expressions so that like terms shall stand in the same column. Find the sum of the terms in each column, and connect the results with their proper signs.*

A polynomial may be regarded as the sum of its monomial terms. Thus, the sum of the terms $(+a)$, $(-b)$, and $(-3c)$ is $a - b - 3c$. Hence, the sum of two or more polynomials whose terms are all unlike is expressed by *writing their terms with their respective signs.* Thus, the sum of $a - b$, $c - d$, and $m + n - x$ is $a - b + c - d + m + n - x$.

Exercise 9.

Find the sum of:

1. $2x + y$, $5x + 3y$, $-3x - 2y$, and $-4x + 3y$.

2. $5x + 3y + 3a$, $-7x + 4y - 8a$, and $2x - 3y$.

3. $3b - 3c$, $2c - 2x$, $3c - 7b$, and $4b - 2c + 3x$.

4. $14a + x$, $13b - y$, $-11a + 2y$, and $x - 2a - 12b$.

5. $ax - 4mn + bd$, $bd - ax - 3mn$, $7mn - 3ax + 3bd$, and $5mn - 30ax - 9bd$.

6. $a - b$, $2b - c$, $2c - d$, $2d - 3a + n$, and $m - n + x$.

7. $abc + 3abm - 5cm$, $3cm + 11abm + 9abc$, $90abm - 21cm - 31abc$, and $3cm - 51abm + 13abc$.

8. $m + n + p$, $m - n - p$, $m - n + p$, and $m + n - p$.

9. $a+b+c+d$, $a+b+c-d$, $a+b-c+d$, $a-b+c+d$, and $-a+b+c+d$.

10. $1.25\,ab + 1.1\,c + .99\,b$, and $3\,ab + 2.2\,c + 1.01\,b$.

11. $\frac{1}{3}a + \frac{1}{2}ab + .9\,b^2$, $\frac{7}{8}a - \frac{2}{13}ab - 15\,b^2$, $\frac{6}{5}a + \frac{5}{2}ab + .6\,b^2$, and $.1\,a - 1.01\,ab$

12. $\frac{4}{3}m^2 - .2\,m + \frac{1}{3}$, $.1\,m^2 + .01\,m - .2$, $m^2 + 3\frac{1}{4}mn - \frac{1}{2}$, and $\frac{1}{2}m - 5\frac{2}{3}mn - 1\frac{7}{8}m^2 - 2\frac{1}{5}$.

13. $xy - ac$, $3\,xy - 9\,ac$, $-7\,xy + 5\,ac + 1.01\,cd$, $4\,xy + 6\,ac - .09\,cd$, and $-xy - 2\,ac + cd$.

14. $.5\,a^3 - 2\,a^2b - \frac{3}{2}b^3$, $\frac{3}{2}a^2b - .75\,ab^2 + 2\,b^3$, and $-\frac{3}{2}a^3 + ab^2 + \frac{1}{2}b^3$.

15. $3\,(m-n)^3 + .3\,(x+y)^5$, $.4\,(m-n)^3 - .2\,(x+y)^5$, $.7\,(m-n)^3 - 3.03\,(x+y)^5$, and $5.1\,(m-n)^3 - 3.1\,(x+y)^5$.

16. $5\,a^3b^2 - 8\,a^2b^3 + x^2y + xy^2$, $4\,a^2b^3 - 7\,a^3b^2 - 3\,xy^2 + 6\,x^2y$, $3\,a^3b^2 + 3\,a^2b^3 - 3\,x^2y + 5\,xy^2$, and $2\,a^2b^3 - a^3b^2 - 3\,x^2y - 3\,xy^2$.

17. $\frac{3}{4}ax^2 - \frac{2}{3}a^2 + \frac{2}{5}x^3y + b^3$, $3\,ax^2 + \frac{5}{4}x^3y + 7.5\,a^2 + \frac{1}{3}b^3$, $2\,ax^2 + \frac{3}{2}x^3y - \frac{1}{2}a^2 - \frac{2}{3}b^3$, and $\frac{1}{16}ax^2 + \frac{1}{8}x^3y + \frac{3}{4}a^2 - \frac{1}{5}b^3$.

18. $ac^2 + \frac{1}{2}ab^2 + \frac{3}{4}a^3 - \frac{3}{2}a^2b - \frac{2}{3}abc + \frac{1}{3}a^2c$, $\frac{1}{2}a^2b + \frac{1}{3}b^3 + \frac{2}{3}ab^2 + \frac{1}{5}bc^2 + 2\,abc + 1\frac{1}{2}b^2c$, and $1.1\,a^2c - 1.2\,ac^2 + \frac{1}{2}b^2c - \frac{3}{2}bc^2 - 1.3\,c^3 + 1.23\,abc$.

19. $3.1x^3 - 4.2\,x^2 + 1.2\,x + 1.7$, $2.22\,x^3 - 1.2\,x^2 + 3.33\,x - 10.09$, $2\,x^3 + 7\,x^2 - 2\,x + 1$, $3\,x^3 + 1.22\,x^2 + 12.12 - 1.33\,x$, and $11.11\,x^3 + 5.55\,x^2 - 6.2\,x + 3.77$.

20. $a^3b^2c^3 + a^2b^3c^2 + 3\,a^2b^3c^3$, $1.8\,a^3b^2c^3 - 1.4\,a^2b^3c^3 + 1.5\,a^3b^3c^2$, $1.5\,a^2b^3c^2 - 1.9\,a^3b^2c^3 + 1.3\,a^2b^3c^3$, and $1.7\,a^3b^2c^3 - 1.2\,a^2b^3c^2 + 1.01\,a^2b^3c^3$.

21. $2c^m + .1a_n + 3bc$, $.5c^m + 3.9a^n + 2.02bc$, $c_m + 2.09a^n - \frac{7}{8}bc$, and $\frac{1}{3}b + \frac{5}{6}bc - 3.03$.

22. $ab - x + \frac{1}{2}xy$, $.1x + .01ax - 2.02ab$, $\frac{1}{2}ax + \frac{1}{3}xy - \frac{4}{5}ab$, $6x - 1.01ax + .1x$, and $\frac{1}{5}xy + \frac{3}{4}ax - \frac{7}{8}xy$.

23. $3a + .1y - 7.01z + 6.01y + .2a + 3z - 1.5a - .8z + 9.01a + 3.03y + z - 4.04y - 2.01z + 2.2a - y$.

24. $3ab + 9 - x^2y$, $x^3y + 3xy + 5$, $6xy^2 + 4x^2y - 3xy$, $10xy + 1 + 3xy^2$, and $17 - 3x^2y - 2x^3y$.

25. $.5(m - 3x)^n$, $-\frac{7}{8}(m - 3x)^n$, $.75(m - 3x)^n$, and $-1.25(m - 3x)^n$.

26. $a^2 + b^4 + c^3$, $-4a^2 - 5c^3$, $8a^2 - 7b^4 + 10c^3$, and $6b^4 - 6c^3$.

27. $3a^2 - 4ab + b^2 + 2a + 3b - 7$, $2a^2 - 4b^2 + 3a - 5b + 8$, $10ab + 8b^2 + 9b$, and $5a^2 - 6ab + 3b^2 + 7a - 7b + 11$.

28. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$, $4x^3y - 12x^2y^2 + 12xy^3 - 4y^4$, $6x^2y^2 - 12xy^3 + 6y^4$, and $4xy^3 - 4y^4$.

29. $a^3 + ab^2 + ac^2 - a^2b - abc - a^2c$, $a^2b + b^3 + bc^2 - ab^2 - b^2c - abc$, and $a^2c + b^2c + c^3 - abc - bc^2 - ac^2$.

30. $5a^3 - 2a^2b + 9ab^2 + 17b^3$, $-2a^3 + 5a^2b - 4ab^2 - 12b^3$, $b^3 - 4ab^2 - 5a^2b - a^3$, and $2a^2b - 2a^3 - 6b^3 - ab^2$.

31. $x^m - y^n + 3a^p$, $2x^m - 3y^n - a$, and $x^m + 4y^n - a^a$.

CHAPTER III.

ALGEBRAIC SUBTRACTION.

19. In Art. 12 it was shown that to subtract a positive number means to count so many units in the *negative* direction, and to subtract a negative number means to count so many units in the *positive* direction. Hence, the addition of a positive number produces the same result as the subtraction of a negative number having the same absolute value.

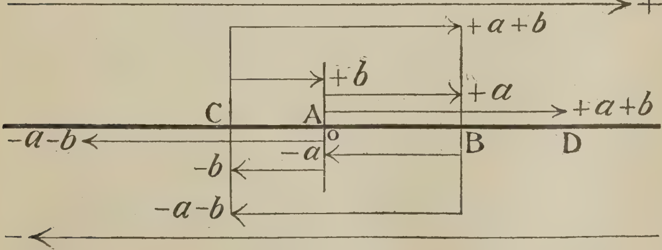
Thus, $+3+(+6)=+3+6=9$. $+3-(-6)=+3+6=9$.

Also, the subtraction of a positive number produces the same result as the addition of a negative number having the same absolute value.

Thus, $+4-(+6)=+4-6=-2$. $+4+(-6)=+4-6=-2$.

We observe that the subtraction of one number from another produces the same result as counting or measuring from the place occupied by the subtrahend to the place occupied by the minuend. Thus,

Subtract $-b$ from $+a$; also $+a$ from $-b$.



Explanation. Suppose the algebraic numbers to be accurately measured on the line of numbers CD. We start at B in the *positive* series, count b units in the *positive* direction, and arrive at D; and the distance from A (0) to D is equal in *absolute* value to AD in the *positive* direction. But in counting from A to D the *absolute* value is the same as the *absolute* value in counting from C (the subtrahend) to B (the minuend), and we have **counted in the direction opposite to that indicated by the sign of the subtrahend.** Thus,

$$CB = +(+b) + (+a) = a + b. \quad \text{That is,}$$
$$(+a) - (-b) = a + b.$$

Subtracting $+a$ from $-b$ gives the same result as counting from a in the positive series to b in the negative series, and the distance from B to C is equal in *absolute* value to BC in the *negative* direction. Thus,

$$BC = +(-a) + (-b) = -a - b. \quad \text{That is,}$$
$$(-b) - (+a) = -b - a, \text{ or } -a - b. \quad \text{Hence,}$$

Algebraic Subtraction is the operation of finding the difference from the subtrahend to the minuend.

To subtract $-5a$ from $+2a$ is the operation of finding *how far and in what direction* we must go to pass from $5a$ in the negative series to $2a$ in the positive series, and is found, by counting from $-5a$ to $+2a$, to be $7a$ *units* in the *positive* direction. That is,

$$+2a - (-5a) = +7a.$$

To subtract $+5a$ from $-2a$, we count from $5a$ in the positive series to $2a$ in the negative series and pass over $7a$ *units* in the *negative* direction. That is,

$$-2a - (+5a) = -7a.$$

These differences may be found by changing the signs of the subtrahend and proceeding as in addition, as shown by a comparison of results. Thus,

Minuend. Subtrahend.		By Addition.
$+2a - (-5a) = +7a$	or,	$+2a + (+5a) = +7a.$
$-2a - (+5a) = -7a$		$-2a + (-5a) = -7a.$
$+a - (-b) = +a + b$		$+a + (+b) = a + b.$
$-b - (+a) = -a - b$		$-b + (-a) = -a - b.$

Hence, in general,

To Subtract one Algebraic Number from another. *Change the sign of the subtrahend, and add the result to the minuend.*

Notes: 1. Algebraic subtraction considered as an operation is not distinct from addition; for it is equivalent to the algebraic addition of a number with the *opposite* algebraic sign. It includes not only distance but *direction*, and direction depends upon the sign of the subtrahend and which number is considered the minuend.

2. Algebraic subtraction is not in all cases diminution. Thus,

$$8 - (-2) = 10; \text{ also } 2 - (-8) = 10.$$

EXAMPLE 1. Subtract $+3a^3bcm^5$ from $+10a^3bcm^5$.

Solution. Changing the sign of the subtrahend, and proceeding as in addition, we have $+10a^3bcm^5 + (-3a^3bcm^5) = +7a^3bcm^5$.

EXAMPLE 2. Subtract $+27(x^2 - y^3)^3$ from $13(x^2 - y^3)^3$.

Solution. Treating $(x^2 - y^3)^3$ as a simple symbol, changing the sign of the subtrahend and proceeding as in addition, we have

$$13(x^2 - y^3)^3 + [-27(x^2 - y^3)^3] = -14(x^2 - y^3)^3.$$

Exercise 10.

From:

1. $+9a^3bc$ take $-a^3bc$; $-14ab^3xy^5$ take $+19ab^3xy^5$.

2. $+x^2y^2$ take $-x^2y^2$; $+99mnp^3rst^{10}$ take $+99mnp^3rst^{10}$.

3. $-10axy$ take $-axy$; xy^3 take $-bc$.

From the sum of:

4. $-11x^5$, $+.5x^5$, and $+1.25x^5$ take $+5.5x^5$.

5. abc^3, $-3abc^3$, and $+.3abc^3$ take the sum of $-abc^3$, $+3.03abc^3$, and $-1.01abc^3$.

6. $108mnp^{10}$, $-10.8mnp^{10}$, and $+mnp^{10}$ take the sum of $-10mnp^{10}$, $+33mnp^{10}$, and $-108.1mnp^{10}$.

7. $5(x+y)$, $-2(x+y)$, and $+(x+y)$, take the sum of $-(x+y)$, $+6(x+y)$, and $-2.5(x+y)$.

Find the aggregate value of:

8. $+17ax^3-(-5ax^3)+(-24ax^3)-(+ax^3)$.

9. $+19axy^2+(+axy^2)-(-5axy^2)$.

10. $+x^2y+(-x^2y)-(+x^2y)-(-x^2y)+(-\frac{1}{2}x^2y)-(+3x^2y)-(-10x^2y)+(-5x^2y)$.

11. $+\frac{1}{3}(a+b)^2-[-.1(a+b)^2]+[-(a+b)^2]-[+\frac{1}{2}(a+b)^2]+10(a+b)^2$.

12. $\frac{2}{3}x^n+(+\frac{1}{5}x^n)+(-.1x^n)-(+\frac{2}{15}x^n)+(-\frac{1}{7}x^n)+(+1.1x^n)-(-3\frac{1}{3}x^n)$.

20. EXAMPLE 1. Subtract $3ab+5a^3y^5-14a^3-7y^3$ from $15a^3-8y^3+23a^3y^5$.

Process.

Minuend	$15a^3-8y^3+23a^3y^5$
Subtrahend, with signs changed	$+14a^3+7y^3-5a^3y^5-3ab$
Difference	$29a^3-y^3+18a^3y^5-3ab$

EXAMPLE 2. Subtract $3xy^2+n-5a^2b+5p^3$ from $5xy^2-3a^2b+3m$.

Process.

Minuend	$5xy^2-3a^2b+3m$
Subtrahend, with signs changed	$-3xy^2+5a^2b-n-5p^3$
Difference	$2xy^2+2a^2b+3m-n-5p^3$

Hence, in general,

To Subtract one Polynomial from another. *Change the algebraic sign of every term in the subtrahend, and add the result to the minuend.*

Note. It is not necessary that the signs of the subtrahend be *actually* changed, we may *conceive* them to be changed.

Exercise 11.

Subtract:

1. $5x - 3y + 2z$ from $3x + y - z$.

2. $x - y + z$ from $-x - 2y - 3z$.

3. $a - b + 20c$ from $-a - b + 10c$.

4. $\frac{1}{2}x - \frac{1}{2}y - \frac{1}{3}z$ from $\frac{3}{4}x + y + z$.

5. $\frac{3}{2}x - \frac{3}{7}y + \frac{1}{10}z$ from $-\frac{1}{2}x + \frac{4}{7}y - \frac{1}{10}z$.

6. $\frac{1}{2}x - \frac{3}{2}y - \frac{1}{6}$ from $-\frac{1}{3}x - \frac{2}{3}y + \frac{1}{6}$.

7. $a^3 - 4a^2b + 2bc + 5$ from $3a^3 - a^2b - bc - 5$.

8. $x^2y - 3abx + 2xy^2 - 1$ from $3x^2y - abx + 2xy^2$.

9. $abcxy + 2aby - 4bx$ from $2abcxy - aby + bx - 3$.

10. $acy - bxy + abc - 1$ from $bxy - 4acy - abc + a$.

11. $.4x^4 - .3x^3 + .2x^2 - 7.1x + 9.9$ from $x^4 - 2.10x^3 + .2x^2 - .07x + .9$

12. $1.2x^5 - 1 + x + 1.1x^4 + 1.7x^3 + a$ from $1 - x - .1x^4 + .2x^5 - .3x^3 + a$.

13. $\frac{1}{2}mn^2 + \frac{5}{6}n^2 - \frac{1}{5}m^3 + \frac{5}{9}n^2$ from $\frac{3}{4}m^3 - \frac{1}{3}mn^2 - n^2$.

14. $.125m^3 - .66\frac{2}{3}mn^2 - .83\frac{1}{3}nm^2 - .66\frac{2}{3}n^3 + a$ from $\frac{1}{8}m^3 - \frac{2}{3}mn^2 - \frac{5}{6}m^2n + \frac{1}{3}n^3$.

15. $\frac{2}{3}m^2 - \frac{3}{5}y - \frac{7}{4}n + \frac{1}{3}x$ from $\frac{4}{5}m^2 - \frac{2}{3}y + \frac{6}{7}n - \frac{1}{3}x$.

16. $a^2bc + xy^n - \frac{2}{5}c$ from $3\frac{1}{2}a^2bc + 2\frac{1}{3}xy^n - 4\frac{2}{3}c$.

17. $10c - a - b + 5d + 6a - 15c + 3d$ from $25a - b - 5c + 8d - 20a$.

18. $x^m - 3x^ny^m - y^m$ from $4x^m + x^ny^m - x^m$.

19. $-.9\,a^m\,x^2 - .3\,a\,b^3\,x + .6 + .03\,b^m\,c\,x^2$ from $.9\,a^m x^2 + 1.3 - 2\,a\,b^3\,x + .4\,b^m\,c\,x^2$.

20. From the sum of $\frac{1}{2}\,a^2 - \frac{1}{3}\,b^3 + \frac{1}{4}\,c^4$, $\frac{4}{3}\,a^2 - 3\frac{1}{4}\,c^4$, and $2\frac{3}{4}\,b^3 - \frac{3}{8}\,a^2 - 1\frac{3}{5}\,c^4$ take $\frac{11}{24}\,a^2 - \frac{7}{12}\,b^3 - 4\,c^4$.

21. From the sum of $3\,x^3 - y^2\,x - x\,z^3$, $7\,x^3 - 1\frac{1}{2}\,x\,z^3 + 11\frac{1}{5}\,y^2\,x$, and $11\,x\,z^3 - 6\frac{1}{4}\,x^3 - 2\,y^2 x$ take $x\,z^3 - 25\,y^2\,x + \frac{1}{5}\,x^3$.

22. From $x^n + y^m$ take the sum of $11\,x^n + y^m - z$, $-6\,x^n - 5\,y^m - 3\,z$, and $-5\,x^n + 3\,y^m + 4\,z$.

23. Add the sum of $3\frac{1}{3}\,y - .3\,y^2$ and $5 - 3\,y + 2.7\,y^3$ to the difference obtained by subtracting $3 + 1\frac{2}{3}\,y^2 - .5\,y$ from $1 - y^3$.

Queries. Why change the signs of the subtrahend in subtracting? Why add the subtrahend, with signs changed, to the minuend? Does the use of the signs + and − in Algebra differ from their use in Arithmetic? How?

Miscellaneous Exercise 12.

1. From $m^3 - n - 1$ take the sum of $2\,n - 3 + 2\,m^3$ and $3\,m^3 - 4 + 5\,n^2 - n$.

2. From the sum of $1 - 8.8\,y + .9\,x^3$ and $1.1\,x^3 + 3\,x^2 - .2\,y - 1$ subtract $2\,x^3 - x^2 + 5\,y$.

3. Take $x^2 + x - 1$ from $2\,x^3$, and add the result to $-2\,x^3 - x^2 - x + 1$.

4. Take $a^2 - b^2$ from $a\,b - b^2$, and add the remainder to the sum of $a\,b - a^2 - 3\,b^2$ and $a^2 + 2\,b^2$.

5. To the sum of $m + n - 3\,p + 5$ and $2\,m + 3\,n + 5\,p - 3$ add the sum of $m - 4\,n - 7\,p$ and $5\,p - 6\,m - 2$.

6. Take $3\,x^{3m} - 2\,x^{2n}\,y^m - y^{m-1}$ from $3\,y^{m-1} + 2\,x^{2n}\,y^m - x^{3n} + x^m$.

7. Take $2\,x^{\frac{1}{2}}\,y^{\frac{1}{2}} - 3\,z^2 + 2\,y^{\frac{2}{3}} + z - 1$ from $x^{\frac{1}{2}}\,y^{\frac{1}{2}} + z - 4\,y^{\frac{2}{3}}$.

8. Take $.8\,b^{\frac{2}{3}}\,y^{\frac{1}{2}} - .4\,a^2\,x^{\frac{1}{2}} + .3\,d$ from $.4\,a^2\,x^{\frac{1}{2}} + .3\,c - 1.2\,b^{\frac{2}{3}}\,y^{\frac{1}{2}}$.

9. Take $\frac{3}{2}\,x^{\frac{2}{3}} - \frac{5}{2}\,x^{\frac{1}{3}}\,y^{\frac{1}{3}} + 33\frac{1}{3}\,y^{\frac{2}{3}}$ from $\frac{2}{3}\,x^{\frac{2}{3}} + \frac{2}{5}\,x^{\frac{1}{3}}\,y^{\frac{1}{3}} + \frac{1}{3}\,y^{\frac{2}{3}}$.

10. From the sum of $.7\,c\,y^{\frac{2}{3}} - .4\,a\,x + .5\,b$, $.04\,b - \frac{2}{5}\,c\,y^{\frac{2}{3}} + \frac{1}{3}\,m$, $-\frac{3}{4}\,a\,x + \frac{5}{2}\,c\,y^{\frac{2}{3}} - \frac{6}{5}$, and $\frac{10}{3}\,a\,x - .23\,b - .8\,m + .3$ take the sum of $.55\,a\,x + \frac{4}{3}\,m + \frac{3}{11}$ and $.33\,m - 1.1\,c\,y^{\frac{2}{3}} + .67\,m$.

11. Find the sum of $a^m - 7\,b^n + c\,p$ and $\frac{3}{2}\,b^n + \frac{9}{4}\,a^m$, and subtract the result from $c\,p - 4\,n$.

12. From $a^m - 2\,x^p - x^n$ take the sum of $\frac{1}{2}\,a^m - \frac{3}{10}\,b^n - x^p$ and $\frac{1}{2}\,a^m + \frac{3}{2}\,b^n - y^n - x^n$.

13. From $-a^m - b^n - c^p - d^q$ take the sum of $\frac{1}{2}\,a^m + \frac{2}{3}\,b^n - \frac{3}{4}\,c^p$, $\frac{3}{15}\,a^m - \frac{11}{40}\,c^p$, and $\frac{11}{3}\,b^n + d^q$.

14. From $3\,(a^3 - b^3)^4 - (x^3 + y^2)^3$ take $\frac{2}{3}\,(x^3 + y^2)^3 - c^5 + 3\frac{1}{3}\,(a^3 - b^3)^4$.

15. From unity take $3\,a^2 - 3\,a + 1$, and add $5\,a^2 - 3\,a$ to the result.

16. Add $3\,x^{2n} - 7\,x^n + 1$ and $3\,x^{3n} + x^n - 3$, and diminish the result by $x^{2n} - 2$.

17. From zero subtract $\frac{3}{4}\,a^3 - \frac{7}{3}\,x + 2$.

18. From $.3\,m^3 - 1 + \frac{1}{2}\,n$ take $5\,n^2 - 2.7\,m^3 - \frac{1}{2}\,n$, then take the difference from zero, and add this last result to $-5\,n^2 + 3.3\frac{1}{3}\,m^3 + n$.

19. What expression must be subtracted from $10y^2 + y - 1$ to leave $3y^2 - 17y + 3$?

20. What expression must be subtracted from $a - 5x + y$ to leave $2a - 5x + y$?

21. From what expression must $a^2 - 5ab - 7bc$ be subtracted to give a remainder $5a^2 + 3ab + 7bc$?

22. From what expression must $a^{\frac{1}{2}} b^{\frac{1}{3}} - b^{\frac{1}{4}} c^{\frac{1}{5}} + 6a^m c^n$ be subtracted to leave a remainder $b^{\frac{1}{4}} c^{\frac{1}{5}} - 6a^m c^n$?

23. To what expression must $\frac{7}{3}a^3 + 2\frac{1}{4}a - 1\frac{1}{5}a^2 - 3$ be added so as to make $2\frac{1}{6}a^3 - 2\frac{1}{4}a + 3\frac{1}{7}a^2 + \frac{5}{8}$?

24. To what expression must $5xy - 11bc - 7mn$ be added to produce zero?

25. What expression must be added to $3x^n - 3x^{n-1} + 2$ to produce $x^n + x^{n-1} - 6$?

26. What expression must be added to $mx^m - x^n + 2$ to produce $mx^m - 2$?

27. From the sum of $.6(x+y)^{\frac{1}{2}} + .3a^n + x^m$, $\frac{3}{5}a^n x^m - c^3 - \frac{6}{5}(x+y)^{\frac{1}{2}}$, and $\frac{3}{4}(x+y)^{\frac{1}{2}} - \frac{3}{8}a^n x^m$, take the sum of $.3(x+y)^{\frac{1}{2}} - \frac{1}{2}a^n x^m$, $\frac{7}{8}a^n x^m - 6.5(x+y)^{\frac{1}{2}} + c^p$, and $\frac{1}{10}(x+y)^{\frac{1}{2}} + 3.3a^n x^m - 3$.

Algebraic Subtraction may be defined as the operation of finding a number which added to a given number, will produce a given sum. The sum is now called the *minuend*, the given number is the *subtrahend*, and the required number is the *difference*.

CHAPTER IV.

ALGEBRAIC MULTIPLICATION.

21. EVIDENTLY $5m \times 6n = 5 \times 6 \times m \times n = 30mn$. Hence, in Algebra, the product is the same in whatever order the factors are written.

$a \times a \times a \times a$ or $a\,a\,a\,a$ is written a^4, and *shows that* a *is taken four times as a factor.* $a \times a \times a \times a \times a$ or $a\,a\,a\,a\,a$ is written a^5, and *shows that* a *is taken five times as a factor.* $a \times a \times a \times \ldots$ to n factors, or $a\,a\,a \ldots$ to n factors is written a^n, and *shows that* a *is taken* n *times as a factor.* Hence,

An **Integral Exponent** shows how many times a number or term is taken as a factor.

a^2 is read *a second power*, or *a exponent two*, or *a square.* a^3 is read *a third power*, or *a exponent three*, or *a cube.* Hence,

A **Power** is the product of two or more equal factors. The degree of the power is indicated by an exponent.

$$a^3 = a\,a\,a,$$

and
$$a^6 = a\,a\,a\,a\,a\,a.$$

Hence,
$$a^6 \times a^3 = a\,a\,a\,a\,a\,a \times a\,a\,a = a^9.$$

$$a^n = a\,a\,a\,a \ldots \text{ to } n \text{ factors,}$$

and
$$a^m = a\,a\,a\,a \ldots \text{ to } m \text{ factors.}$$

Multiplying the second expression by the first, we have,

$$a^m \times a^n = a\,a\,a \ldots \text{ to } m \text{ factors} \times a\,a\,a \ldots \text{ to } n \text{ factors}$$
$$= a\,a\,a \ldots \text{ to } (m+n) \text{ factors}$$
$$= a^{m+n}.$$ In which m and n are *any* numbers

whatever. Similarly for the product of more than two powers of a factor. Hence,

The powers of a number are multiplied by adding the exponents.

If the multiplicand and multiplier consist of powers of different factors, we use a similar process. Thus,

$$\begin{aligned}3m^5 \times 2m^3n^2 \times 5m^2n^3 &= 3 \times 2 \times 5\, m\,m\,m\,m\,m\,m\,m\,m\,m\,m \\ &\quad \times n\,n\,n\,n\,n \\ &= 30\, m^{10}\, n^5.\end{aligned}$$

$$\begin{aligned}a^nb^m \times a^pb^r &= a\,a\,a \ldots \text{to } n \text{ factors} \times a\,a\,a \ldots \text{to } p \text{ factors} \\ &\quad \times b\,b\,b \ldots \text{to } m \text{ factors} \times b\,b\,b \ldots \text{to } r \text{ factors} \\ &= a\,a\,a \ldots \text{to } (n+p) \text{ factors} \times b\,b\,b \ldots \text{to } (m+r) \\ &\quad \text{factors} \\ &= a^{n+p}\, b^{m+r}.\end{aligned}$$

Hence, in general,

To Find the Product of Two or more Monomials. *To the product of the numerical coefficients annex the factors, each taken with an exponent equal to the sum of the exponents of that factor.*

Notes: 1. When no exponent is written, the exponent is 1. Thus, a is the same as a^1, b as b^1.

The *exponent is used to save repetition.*

2. We read a^2, *a square*, and a^3, *a cube*, because if a represents the number of units of length in the side of a square, and the edge of a cube, then a^2 and a^3 will represent the number of units in the *surface* and *volume* of the square and cube, respectively.

Illustrations.

$11\, m^{19} \times 10\, m^{10} = 11 \times 10\, m^{19+10} = 110\, m^{29}.$

$$\begin{aligned}3a^2bcm \times 2ab^2cm \times 5abc^2m^2 &= 3 \times 2 \times 5\, a^{2+1+1}b^{1+2+1}c^{1+1+2}m^{1+1+2} \\ &= 30\, a^4\, b^4\, c^4\, m^4.\end{aligned}$$

$3\, a^{\frac{2}{3}}\, b^{\frac{3}{4}}\, c^3 \times 4\, a \times b^{\frac{1}{4}}\, c^{\frac{1}{5}} = 3 \times 4\, a^{\frac{2}{3}+1}\, b^{\frac{3}{4}+\frac{1}{4}}\, c^{3+\frac{1}{5}} = 12\, a^{\frac{5}{3}}\, b\, c^{\frac{16}{5}}.$

$2^{\frac{1}{2}}\, x^2y^n \times 2^{\frac{1}{2}}x^{-3} \times x^5\, y^n = 2^{\frac{1}{2}+\frac{1}{2}}\, x^{2-3+5}\, y^{n+n} = 2\, x^4\, y^{2n}.$

Exercise 13.

Find the product of:

1. x^3 and $7x^5$; $3ax$ and $5a^2x^5$; a^3bx^3 and $2a^2b^3x^2$.

2. $\frac{5}{9}xyz^{10}$ and $\frac{3}{5}x^9y^9mn$; $\frac{2}{3}abcdm^2n^2$ and $\frac{1}{3}a^2b^2c^2d^2mn$.

3. $\frac{2}{3}b^2c^3y$ and $\frac{3}{2}ab^3c^5y^{n+1}$; $\frac{3}{4}a^3x^6y^7$ and $\frac{4}{7}a^{15}x^3y^4z$.

4. $3ax^3y^3$ and $10a^{10}xy^{10}$; $\frac{1}{5}x^my^n$ and $\frac{5}{4}x^1y^1$.

5. $3abcx^{10}y$ and $\frac{2}{3}a^nb^9c^9y^{10}$; $\frac{1}{4}a^7b^6c^5xy$ and $\frac{2}{5}ab^{10}c^9x^ny^m$.

6. $\frac{2}{7}a^mb^nx^ry^s$ and $.2a^2b^3x^4y^5$; x^3y^4 and x^6y^5.

7. a^mb^n and a^nb^r; $x^{n+1}y^{m+1}$ and $5.7x^{-1}y^{-1}$.

8. $abxy^5$ and $a^2b^2x^5y$; $a^{m+p}b^{n+r}$ and $a^{m-p}b^{n-r}$.

9. $.55x^{-p+q}y^{-r+s}$ and $.5x^{p+q}y^{r+s}$; $.3a^{2-m}x^{3-n}$ and a^mx^n; $5a^{-\frac{1}{4}}b^nx^r$ and $5a^{\frac{1}{2}}bx^{-2r}$.

10. $2a^3x$, xy^2, a^2y, $a^3x^3y^3$, and ay^5.

11. a^m, b^n, $3c^p$, a^p, b^q, c^m, and d^q.

12. $2c^md$, ac^n, d^2x^n, a^mx^m, and $c^{\frac{1}{2}}$.

13. $a^{\frac{3}{4}}mx^{\frac{1}{2}}$, $a^{\frac{1}{2}}n^{\frac{3}{2}}x^{\frac{2}{3}}$, $amxy$, and $2a^2n^3x^4y^5$.

14. $a^{\frac{1}{3}}x^{\frac{5}{6}}$, $a^{\frac{2}{5}}y^{\frac{1}{2}}$, $a^{\frac{2}{3}}x^{-\frac{1}{3}}$, $a^{\frac{1}{7}}y^{-\frac{3}{4}}$, $5^{\frac{3}{4}}a^{-\frac{3}{4}}x^{\frac{5}{9}}$, and $5^{\frac{1}{4}}x^1y^1$.

15. $\frac{3}{4}a^3mx^n$, $\frac{2}{3}m^{\frac{2}{3}}x^{\frac{3}{5}}$, $.3ax^{-2}$, $5.1m^s$, and $a^{-2}x^{-r}$.

16. $3y^n$, $a^{-5}y^2z^m$, a^rb^n, a^3b^c, $\frac{3}{4}a^xy^n$, and $\frac{5}{6}a^xb^{-2n}$.

17. $(a+b)$, $5(a+b)^2$, $3(a+b)^4$, $\frac{1}{2}(a+b)^5$, and $(a+b)$

18. $(a+b)(c+d)^5$, $(a+b)^2$, $3(c+d)^3$, and $(a+b)^7(c+d)^2$.

19. $3(a+b)^n(x-y)^m$, $\frac{1}{5}(a+b)^3$, and $\frac{5}{6}(x-y)^5$.

22. Algebraic Multiplication is the operation of adding as many numbers, each equal to the multiplicand, as there are units in a positive multiplier; it is also the operation of subtracting as many numbers, each equal to the multiplicand, as there are units in a negative multiplier. Hence,

The multiplier shows that the multiplicand is taken so many times to be added, or so many times to be subtracted.

Thus,

$$(+6) \times +4 = (+6) + (+6) + (+6) + (+6) = +(+24) = +24;$$
$$(-6) \times +4 = (-6) + (-6) + (-6) + (-6) = +(-24) = -24;$$
$$(+6) \times -4 = -(+6) - (+6) - (+6) - (+6) = -(+24) = -24;$$
$$(-6) \times -4 = -(-6) - (-6) - (-6) - (-6) = -(-24) = +24.$$

The sign of the multiplicand (6) shows that the product (24) is in the positive and negative series of numbers, respectively; and the sign of the multiplier (4) shows that the first two products are to be *added* and the last two are to be *subtracted*. Hence,

The sign of the multiplicand shows what series of numbers the product is in, and the sign of the multiplier shows what is to be done with the product.

Law of Signs. *The product of two factors is positive when the factors have like signs, and negative when they have unlike signs.*

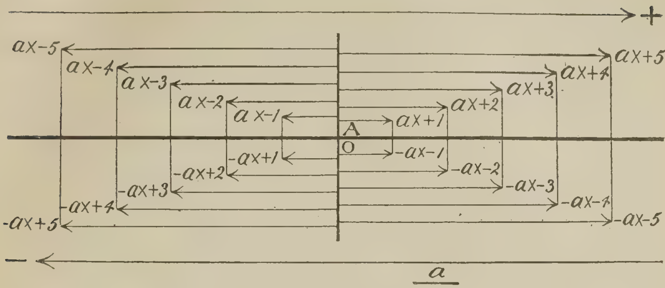
Since

$$-2 \times -2 = +4;\ -2 \times -2 \times -3 = +4 \times -3 = -12;$$
$$-2 \times -2 \times -3 \times -4 = -12 \times -4 = +48;$$
$$-2 \times -2 \times -3 \times -4 \times -5 = +48 \times -5 = -240;$$

etc. Hence,

The product of an even number of negative factors is positive; of an odd number, negative.

The change of signs may be illustrated as follows :



Let the measuring unit be represented by a.

From A (o), the starting-point on the scale, measure toward the right and left. The products of $+a$ and $-a$ by the factors from $+5$ to -5 are :

$$a \times +5,\ a \times +4,\ a \times +3,\ a \times +2,\ a \times +1;$$
$$a \times -1,\ a \times -2,\ a \times -3,\ a \times -4,\ a \times -5;$$
$$-a \times +5,\ -a \times +4,\ -a \times +3,\ -a \times +2,\ -a \times +1;$$
$$-a \times -1,\ -a \times -2,\ -a \times -3,\ -a \times -4,\ -a \times -5;$$

respectively. The directions taken by the products are shown in the figure.

Illustrations.

$$x^2y^3 \times -x^3z \times -\tfrac{6}{5}yz^2 \times -\tfrac{1}{2}xz^5 \times -4yz^2 = +\tfrac{6}{5} \times \tfrac{1}{2} \times 4x^6y^5z^{10}$$
$$= \tfrac{12}{5}x^6y^5z^{10}.$$
$$x^my^{-n} \times -\tfrac{3}{4}x^ny^nz \times -yz^{-r} \times -x^{-2n} = -x^{m+n-2n}y^{-n+n+1}z^{1-r}$$
$$= -x^{m-n}yz^{1-r}.$$

Exercise 14.

Find the product of:

1. $5a$, $-3b$, $7c$, $-2a^2$, $-11a^3$, and a; a^2x, $-ay^2$, ax^2, and $-xy$.

2. abx, $-ay^2$, $-ax$, and a^2x^3; $-ab^3$, $-bc^2$, $-cd^2$, $-a$, $-a^2$, $-a^3$, and $-5a^4$.

3. $-a$, bc, -1, $\frac{1}{5}$, $\frac{3}{15}a^2$, $\frac{4}{3}xy$, and $75a$; ax, cx, $-mx$, $-y^n$, and $.3m$.

4. $\frac{1}{2}abc$, $-d$, ax, -1, and $\frac{3}{2}axyz$; $a^n x_m$, $x^m y^n$, $a^n b^r$, and ab.

5. $-a^2x$, $3x$, ab^2, ay, az, and $axyvw$; axy, $-\frac{3}{10}a^m b^n$, and $-3\frac{1}{3}a^n b^r x^m y^n$.

6. $-a^2bc$, $2b^2cd^2$, $-.5a^3cd^5$, $-\frac{3}{10}a^{-10}b^{-10}c^{10}d^{-10}$, and ab^5d^4.

7. -1, a^{-3}, x^7, ax^{-5}, $a^{10}x^{-3}$, $a^{-5}b^{-6}x^3$, and $-b^4d^2$.

8. ax^2, $-a^2$, -1, $.3ax$, and $-a^{\frac{1}{2}}y^{\frac{1}{6}}$; a^2x, $-a^{\frac{2}{3}}$, $a^{\frac{3}{2}}$, and $-a^{\frac{1}{2}}x^{\frac{1}{2}}$.

9. $-my$, mx, $-mn$, $-xy$, and $x^{\frac{2}{3}}y^{\frac{1}{3}}$; $3a^{\frac{1}{2}}b^{\frac{1}{3}}$ and $-.7a^{\frac{1}{4}}b^{\frac{1}{3}}$.

10. a^n, a^{-n}, a^{3n}, a^{2n}, and a^{5n}. Express the result in two ways.

11. $2^3y^{-r}x^{-1}$, mx^2y^{2r}, $-3^nq^nx^{-1}$, and $-2^{-1}px^3y^r$.

12. 3^{2n}, $-2^{3a}\times 3^{3a}$, 3^{2a}, $-2^{3n}\times 3^{4a}$, $3^{5n}\times 2^a$, and $-2^{6n}\times 3^{a+1}$.

23. EXAMPLE. Multiply $a+b$ by m; also $a-b$ by m.

The symbol $(a+b)m$ means that m is to be taken $(a+b)$ times. Hence,

Process.

$$\begin{aligned}(a+b)m &= m+m+m+\ldots\text{ taken } a+b \text{ times}\\ &= (m+m+m+\ldots\text{ taken } a \text{ times}) + (m+m+m+\ldots\text{ taken } b \text{ times})\\ &= am+bm. \qquad (1)\end{aligned}$$

Also,

$$(a-b)\,m = m+m+m+\dots \text{ taken } a-b \text{ times}$$
$$= (m+m+m+\dots \text{ taken } a \text{ times}) - (m+m+m+\dots \text{ taken } b \text{ times})$$
$$= (m \times a) - (m \times b)$$
$$= am - bm. \qquad (2)$$

Similarly, $(a+b-c)\,m = am + bm - cm.$

These results are obtained by multiplying each term of the multiplicand separately by the multiplier. Hence, in general,

To Multiply a Polynomial by a Monomial. *Multiply each term of the multiplicand by the multiplier, and add the results.*

Exercise 15.

Multiply:

1. $bc+ac-ab$ by abc; $8a^2b^4-\frac{2}{3}b^2c^5-\frac{3}{2}c^3$ by $\frac{1}{12}a^5b^{10}$.

2. $5a^2-b^2-2c^2$ by $a^2b^9c^{10}$; $.6x^3-.5x^3y^3-.3x^4y^9-.2x^5$ by $.2x^2y^3$.

3. $\frac{2}{3}m^2-\frac{1}{3}mn+\frac{2}{3}n^2$ by $\frac{9}{2}mn$; $x-y-\frac{3}{2}x^2y^2$ by xy.

4. $\frac{2}{5}a-\frac{3}{10}b^2+\frac{4}{15}ab^2$ by $\frac{5}{3}ab^2$; $a^7-a^2b^2-ab$ by ab^2.

5. $6a^2a^3-.5a^2b^3x^{10}+.2b^2x^3$ by $\frac{3}{2}abx^3$; px^m-qx^n-r by px^mr.

6. $3a^{m-1}-2b^{n-2}+4a^mb^n$ by ab^2; $.4a^{m-p}b^{3p}-\frac{4}{3}a^{-2p}b^p+b^{2p}$ by $\frac{3}{2}a^{m+p}b^p$.

7. $a^{2m}-3a^mb^{-n}+b^n$ by a^mb^{2n}; $2^{\frac{1}{2}}x^{\frac{1}{2}}-2^{\frac{1}{2}}y^{\frac{1}{2}}+2^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{1}{2}}$ by $2^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{1}{2}}$.

8. $a^{\frac{7}{3}}-a^2b^{\frac{2}{3}}-a^{\frac{1}{3}}b+b^{\frac{1}{3}}$ by $a^{\frac{1}{3}}b^{\frac{2}{3}}$; $x^{\frac{9}{4}}-x^{\frac{2}{3}}y^{\frac{3}{4}}+x^{\frac{3}{4}}y^{\frac{3}{2}}-.5y^{\frac{9}{4}}$ by $x^{\frac{3}{4}}y^{-\frac{1}{2}}$.

Find the product of:

9. $x^2y^1 - 4\,x^1y^3 + 4\,y^4$, $x^1\,y^1$, and $-\,2\,y^2$; $m^{2p} - 2\,m^{2p}n^{2q} + n^{2q}$, m^{-p}, n^{-q}, and $m^p\,n^q$.

10. $\frac{1}{4}\,a^2b^{-4} + \frac{2}{3}\,a\,b^{-3}x + \frac{4}{9}\,b^2x^2$, $\frac{1}{2}\,a\,b^2$, $\frac{2}{3}\,b^{-1}x$, and $\frac{1}{3}\,a^2b^3x^5$.

11. $x^{\frac{2}{3}} - y^{\frac{2}{3}}$, $x^{\frac{1}{3}}$, $y^{\frac{1}{3}}$, and $-\,x^{\frac{2}{3}}\,y^{\frac{2}{3}}$; $a - b^{\frac{4}{3}}$, $a^{\frac{1}{2}}$, $b^{\frac{2}{3}}$, $a^{\frac{1}{2}}\,b^{\frac{1}{2}}$, $-\,a^{\frac{1}{2}}\,b^{\frac{2}{3}}$, and $-\,a\,b^{\frac{4}{3}}$.

12. $x^3 - y^3$, $x^{\frac{9}{4}}$, $x^{\frac{3}{2}}\,y^{\frac{3}{4}}$, $-\,x^{\frac{3}{4}}\,y^{\frac{3}{2}}$, $\frac{2}{3}\,y^{\frac{9}{4}}$, $\frac{5}{3}\,x^{\frac{3}{4}}$, $-\,\frac{9}{2}\,y^{\frac{3}{4}}$, and $x^{\frac{21}{4}}\,y^{\frac{21}{4}}$.

13. $x^{\frac{5}{2}} - .2\,b^{\frac{2}{3}}\,x^2 + .3\,b\,x^{\frac{1}{2}} - b^{\frac{5}{3}}$, $b\,x^{\frac{1}{2}}$, $-\,b^{\frac{2}{3}}\,x^2$, and $\frac{1}{3}\,b^{\frac{5}{3}}\,x^{\frac{5}{2}}$.

14. $\frac{9}{5}\,a^{-2m}$ $\frac{12}{7}\,a^{-m}\,b^{-m} + \frac{4}{3}\,b$, $3\,a^{-m}$, $-\,5\,b^{-m}$, and $7\,a^{-m}\,b^{-m}$.

15. $a^{m+n} + a^n\,b^m + a^m\,b^n + b^{m+n}$, a^m, b^m, a^{-n}, b^{-n}, and $a^{-m}\,a^{-n}\,b^{-m}\,b^{-n}$.

24. EXAMPLE 1. Multiply $m + n$ by $x + y$; also $m + n$ by $x - y$. $(m + n)\,(x + y)$ means that $x + y$ is to be taken $m + n$ times. Hence,

Process

$$
\begin{aligned}
(m+n) \times (x+y) &= (x+y) + (x+y) + (x+y) + \ldots\ldots \text{ taken } m+n \text{ times} \\
&= [(x+y) + (x+y) + (x+y) + \ldots\ldots \text{ taken } m \text{ times}] \\
&\quad + [(x+y) + (x+y) + (x+y) + \ldots\ldots \text{ taken } n \text{ times}] \\
&= (x+y)\,m + (x+y)\,n \\
&= (1) \text{ Art. } 23,\ m\,x + m\,y + n\,x + n\,y. \qquad (1)
\end{aligned}
$$

Also,

$$
\begin{aligned}
(m+n)\,(x-y) &= (x-y) + (x-y) + (x-y) + \ldots\ldots \text{ taken } m+n \text{ times} \\
&= [(x-y) + (x-y) + (x-y) + \ldots\ldots \text{ taken } m \text{ times}] \\
&\quad + [(x-y) + (x-y) + (x-y) + \ldots\ldots \text{ taken } n \text{ times}] \\
&= (x-y)\,m + (x-y)\,n \\
&= (2) \text{ Art. } 23,\ m\,x - m\,y + n\,x - n\,y. \qquad (2)
\end{aligned}
$$

Similarly, $(m + n + p)\,(x + y - z) = m\,x + m\,y - m\,z + n\,x + n\,y - n\,z + p\,x + p\,y - p\,z$

These results are obtained by multiplying each term of the multiplicand separately by each term of the multiplier, and connecting the products with their proper signs.

EXAMPLE 2. Multiply $x^6 - x^5 + 2x^2 - x - 5$ by $x^4 + 3x^3 + 5$.

Process.

$$
\begin{array}{l}
x^6 - x^5 + 2x^2 - x - 5 \\
x^4 + 3x^3 + 5 \\
\hline
x^{10} - x^9 + 2x^6 - x^5 - 5x^4 \\
\quad + 3x^9 \qquad + 6x^5 - 3x^4 - 3x^8 - 15x^8 \\
\qquad\quad + 5x^6 - 5x^5 \qquad\qquad\qquad\qquad + 10x^2 - 5x - 25 \\
\hline
x^{10} + 2x^9 + 7x^6 \qquad - 8x^4 - 3x^8 - 15x^3 + 10x^2 - 5x - 25
\end{array}
$$

Explanation. Multiplying each term of the multiplicand by each term of the multiplier and connecting these results with their proper signs, we have $x^{10} - x^9 + 2x^6 - x^5 - 5x^4 + 3x^9 - 3x^8 + 6x^5 - 3x^4 - 15x^3 + 5x^6 - 5x^5 + 10x^2 - 5x - 25$. *Uniting like terms*, for a *simplified product*, we have $x^{10} + 2x^9 - 3x^8 + 7x^6 - 8x^4 - 15x^3 + 10x^2 - 5x - 25$.

The process used in practice is shown above. The first line under the multiplier contains the product of the multiplicand and x^4. The second contains the product of the multiplicand and $3x^3$. Etc. To facilitate adding, write the several products so that like terms shall stand in the same column.

Note. It is convenient to arrange the terms of the multiplicand and multiplier according to powers of some common letter, ascending or descending.

EXAMPLE 3. Multiply $\frac{2}{3}ax + \frac{2}{3}x^2 + \frac{1}{3}a^2$ by $\frac{3}{4}a^2 + \frac{3}{2}x^2 - \frac{3}{2}ax$.

Solution. Arrange the expressions according to the descending powers of x. Taking the multiplicand $\frac{3}{2}x^2$ times, we have $x^4 + ax^3 + \frac{1}{2}a^2x^2$. Taking it $-\frac{3}{2}ax$ times, and writing the product so that like terms shall stand in the same column, we have $-ax^3 - a^2x^2 - \frac{1}{2}a^3x$. Again, taking it $\frac{3}{4}a^2$ times, and writing the product as before, we have $\frac{1}{2}a^2x^2 + \frac{1}{2}a^3x + \frac{1}{4}a^4$. Adding the partial products, we have $x^4 + \frac{1}{4}a^4$, or arranging alphabetically, $\frac{1}{4}a^4 + x^4$

Process.

$$\begin{array}{l}
\frac{2}{3}x^2 + \frac{2}{3}ax + \frac{1}{3}a^2 \\
\frac{3}{2}x^2 - \frac{3}{2}ax + \frac{3}{4}a^2 \\
\hline
x^4 + ax^3 + \frac{1}{2}a^2x^2 \\
\qquad - ax^3 - a^2x^2 - \frac{1}{2}a^3x \\
\qquad\qquad + \frac{1}{2}a^2x^2 + \frac{1}{2}a^3x + \frac{1}{4}a^4 \\
\hline
x^4 \qquad\qquad\qquad\qquad + \frac{1}{4}a^4
\end{array}$$

EXAMPLE 4. Multiply $-3x^{m+2}y^n - .3x^{m+1}y^{n+1} + 3.3x^my^{n+2}$ by $-.2x^my^{n-2} + 4x^{m-1}y^{n-1}$.

Process.

$$\begin{array}{l}
3.3x^my^{n+2} - .3x^{m+1}y^{n+1} - 3x^{m+2}y^n \\
4x^{m-1}y^{n-1} - .2x^my^{n-2} \\
\hline
13.2x^{2m-1}y^{2n+1} - 1.2x^{2m}y^{2n} - 12.00x^{2m+1}y^{2n-1} \\
\qquad - .66x^{2m}y^{2n} + .06x^{2m+1}y^{2n-1} + .6x^{2m+2}y^{2n-2} \\
\hline
13.2x^{2m-1}y^{2n+1} - 1.86x^{2m}y^{2n} - 11.94x^{2m+1}y^{2n-1} + .6x^{2m+2}y^{2n-2}
\end{array}$$

Explanation. Arrange according to the ascending powers of x, as shown. The product of the multiplicand by $4x^{m-1}y^{n-1}$ gives the first partial product, as shown on the first line under the multiplier. The product of the multiplicand by $-.2x^my^{n-2}$ gives the second partial product. Taking the sum of the partial products, we have the product required. Hence, in general,

To find the Product of two Polynomials. *Multiply the multiplicand by each term of the multiplier, and add the partial products.*

Exercise 16.

Arrange the terms according to the powers of some common letter, and multiply:

1. $a^2 + b^2 - ab$ by $ab + b^2 + a^2$; $a^2 - 2ax + 4x^2$ by $a^2 + 4x^2 + 2ax$.

2. $x^4 + y^4 - x^2y^2$ by $x^2 + y^2$; $x + y + x - y$ by $x + y - x + y$.

3. $y-3+y^2$ by $y-9+y^2$; $a^2y-az+y^3-a^3$ by $y+a$.

4. $\frac{1}{2}x^2-\frac{2}{3}x-\frac{3}{4}$ by $\frac{1}{2}x^2+\frac{2}{3}x-\frac{3}{4}$; $1.6\,a^2+1.2\,ab+9\,b^2$ by $.4\,a-.3\,b$.

5. x^2-y^2+x-y by x^2+y^2+x-y; $\frac{3}{2}x^2-ax-\frac{2}{3}a^2$ by $\frac{3}{4}x^2-\frac{1}{2}ax+\frac{1}{3}a^2$.

6. $x^2-xy+x+y^2+y+1$ by $x+y-1$; $\frac{2}{3}x^2+3\,ax+\frac{1}{5}a^2$ by $2\,x^3+ax-\frac{1}{4}a^3$.

7. $13\,x^3-5\,x^4-x^2+x^5-x+2$ by $x^2-2\,x-2$.

8. $3\,a^2-2\,a^3-2\,a+1+a^4$ by $3\,a^2+2\,a^3+2\,a+1+a^4$.

9. $1.5\,x^3+1.5\,x^2+.5\,x^4+.5\,x+x^5+1$ by $x^2-.5\,x+1+x^4-.5\,x^3$.

10. $1+9\,a+5\,a^3+3\,a^4+7\,a^2+a^5$ by $4\,a^2-3\,a^3+a^4+4-4\,a$.

11. $4\,x^2y^2+8\,xy^3+16\,y^4+2\,x^3y+x^4$ by $x-2\,y$.

12. $x^{12}-x^3y^6+x^6y^4-x^9y^2+y^8$ by z^2+x^3; $x^2y^2-x^3y-xy^3+x^4+y^4$ by $x+y$.

13. $a^2+b^2+c^2-ab-ac-bc$ by $a+b+c$.

14. $a^2+b^2+c^2+bc+ac-ab$ by $a+b-c$.

15. $ab+cd+ac+bd$ by $ab+cd-ac-bd$.

16. $x^2+y^2-3\,x^2-y^2$ by $2\,x+2\,y-2\,(x-y)$.

17. $3\,(m+n)-.1\times(a+b)$ by $a-b+.1\,(m-n)$.

18. ax^m+bx^n+r by ax^m+bx^n+r; $x^{\frac{1}{3}}+y^{\frac{1}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$.

19. a^m+b^n by a^m+b^n; a^m+b^n by a^m-b^n; x^2+b by $x^{\frac{1}{2}}+b^{\frac{2}{3}}$.

20. $3\,x^{m-1} - 2\,y^{n-2}$ by $2\,x - 3\,y^2$; $a\,x^m + b\,x^n + a\,b\,x$ by $a\,x^2 - b\,x^3 - 1$.

21. $3\,a^{2m}x + 3\,a^2\,y + a^{2n}$ by $a^m - a^n + x$; $x^{\frac{1}{3}} - y^{-\frac{2}{3}}$ by $x^2 - y$.

22. $.2\,a^{\frac{1}{2}} - .3\,b^{\frac{2}{3}}$ by $.2\,a^{\frac{1}{2}} + .3\,b^{\frac{2}{3}}$; $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

23. $x^{\frac{3}{4}}\,y^{-\frac{3}{2}} + y^{-\frac{9}{4}} + x^{\frac{8}{2}}\,y^{-\frac{3}{4}} + x^{\frac{9}{4}}$ by $x^{\frac{3}{4}} - y^{-\frac{3}{4}}$.

Find the product of:

24. $1 + x$, $1 + x^4$, and $1 + x^2 - x - x^3$.

25. $x - 2\,a$, $x - a$, $x + a$, and $x + 2\,a$.

26. $3\,x + 2$, $2\,x - 3$, $5\,x - 4$, and $4\,x - 5$.

27. $x^2 - x + 1$, $x^2 + x + 1$, and $x^4 - x^2 + 1$.

28. $x^2 - a\,x + a^2$, $x^2 + a\,x + a^2$, and $x^4 - a^2\,x^2 + a^4$.

29. $a + b$, $a - b$, $3\,a + b$, and $a^3 - 2\,a^2\,b - a\,b^2 + b^3$.

30. $a^m + b^n$, $a^m - b^n$, $a^{2m} + a^m b^n + b^{2n}$, and $a^{2m} - a^m b^n + b^{2n}$.

25. A **Binomial** is a compound expression of *two* terms; as, $a - b$; $a\,b + 2\,b^2$.

In each of the following products, observe that:

$$\begin{array}{rrr}
2\,x^{} & +\ 3 & \\
2\,x^{} & +\ 5 & \\
\hline
4\,x^2 & +\ 6\,x & \\
 & 10\,x & +\ 15 \\
\hline
4\,x^2 & +\ 16\,x & +\ 15
\end{array}
\qquad\qquad
\begin{array}{rrr}
2\,x^{} & +\ 3 & \\
2\,x^{} & -\ 5 & \\
\hline
4\,x^2 & +\ 6\,x & \\
 & -\ 10\,x & -\ 15 \\
\hline
4\,x^2 & -\ 4\,x & -\ 15
\end{array}$$

$$\begin{array}{rrr}
2\,x^{} & -\ 3 & \\
2\,x^{} & +\ 5 & \\
\hline
4\,x^2 & -\ 6\,x & \\
 & 10\,x & -\ 15 \\
\hline
4\,x^2 & +\ 4\,x & -\ 15
\end{array}
\qquad\qquad
\begin{array}{rrr}
2\,x^{} & -\ 3 & \\
2\,x^{} & -\ 5 & \\
\hline
4\,x^2 & -\ 6\,x & \\
 & -\ 10\,x & +\ 15 \\
\hline
4\,x^2 & -\ 16\,x & +\ 15
\end{array}$$

I. The *first term* is the common algebraic term of the binomials multiplied by itself, or *the square of the common algebraic term.*

II. The *second term* is the algebraic sum of the other two terms of the binomial expressions multiplied by the common algebraic term.

III. The *last term* is the algebraic product of the terms which are not common to the binomial expressions. Hence,

To find the Product of two Binomials, having one Common Algebraic Term. *Add together the square of the common term, the algebraic sum of the other two terms multiplied by the common term, and the algebraic product of the terms which are not common.*

In general, $(x+a)(x \pm b) = x^2 + (a \pm b)x \pm ab$ (1)

$$(x-a)(x \pm b) = x^2 + (-a \pm b)x \mp ab \qquad (2)$$

In which a, b, and x represent any numbers.

Notes: 1. It is of the utmost importance that the student should learn to write the products of binomial expressions rapidly, **by inspection**.

2. To square a monomial, *multiply the numerical coefficient by itself, and multiply the exponent of each letter by two.* The proof is evident. Thus, the square of $2a^{\frac{1}{2}}b^n = 2 \times 2 a^{\frac{1}{2}\times 2} b^{n\times 2} = 4ab^{2n}$.

Also, $(3b^{-n}x^m)^2 = 3 \times 3 b^{-n\times 2}x^{m\times 2} = 9b^{-2n}x^{2m}$.

Examples. Write the product of the following by inspection :

$$(2x+7y)(2x-5y);\ (a-9b)(a-8b);\ (a-6)(a+13).$$

Solution. Squaring the common term, we have $4x^2$. Taking the algebraic sum of the other two terms, $+7y$ and $-5y$, we have $+2y$. Multiplying this sum by $2x$, we have $+4xy$. Taking the algebraic product of the terms not common, $+7y$ and $-5y$, we have $-35y^2$. We thus obtain $4x^2+4xy-35y^2$ for the product.

Similarly, $(a-9b)(a-8b) = a^2 + (-9b-8b)\times a + (-9b)\times(-8b)$
$= a^2 - 17ab + 72b^2$.

Also, $(a-6)(a+13) = a^2 + (-6+13)\times a + (-6)\times(+13)$
$= a^2 + 7a - 78$.

Exercise 17.

Write, by inspection, the products of the following:

1. $(a-3)(a+5)$; $(b+6)(b-5)$; $(x+4)(x+3)$; $(x-4)(x+1)$; $(x-7)(x+2)$.

2. $(x-8)(x-6)$; $(a+9)(a-5)$; $(a-8)(a+4)$; $(2x-4)(2x-5)$; $(3x+7)(3x-5)$.

3. $(x^3-3y^2)(x^3-4y^2)$; $(x-7y)(x+8y)$; $(a^m-1)(a^m+2)$; $(3x^5-5)(3x^5-4)$.

4. $(2a^2y^3+4)(2a^2y^3-8)$; $(3ax-4)(3ax+7)$; $(x^3+3a)(x^3-4a)$; $(x^5-3a^2)(x^5+2a^2)$.

5. $(2x+a)(2x-2a)$; $(2x^n+5a)(2x^n-3a)$; $(3x-2y)(3x+y)$; $(-6m+2x^3)(4m+2x^3)$.

6. $(x-a)(x-5a)$; $(a-5b)(a+8b)$; $(a^3-2x)(a^3-6x)$; $(5x^{10}+3a^2)(5x^{10}-4a^2)$.

7. $(3y^2-5x^5)(2y^2-5x^5)$; $(3a^5+2ab)(3a^5-4ab^4)$; $(a^n+3)(a^n-b)$.

8. $(4a+b)(4a-c)$; $(2b-5a)(2c-5a)$; $(ay+\frac{1}{9}x)(ay+\frac{1}{3}x)$; $(a^{\frac{3}{4}}-1)(a^{\frac{3}{4}}+\frac{5}{4})$.

9. $(2x^{\frac{1}{2}}+1)(2x^{\frac{1}{2}}+12)$; $(2a^{\frac{1}{2}}-3ax)(2a^{\frac{1}{2}}+b)$; $(x-.3x^{\frac{1}{2}}y^n)(y-.3x^{\frac{1}{2}}y^n)$.

26. $(x+y)(x-y)=x^2+(y-y)\times x+(+y)\times(-y)$ $=x^2-y^2$. In which x and y *represent any two numbers.* Hence, in general,

To find the Product of tl.. Sum and Difference of two Numbers. *Take the difference of their squares.*

EXAMPLES. Find the product of $(2a^m + 3b^{-n})(2a^m - 3b^{-n})$; $(8p^4 + 11z^{\frac{1}{2}})(8p^4 - 11z^{\frac{1}{2}})$.

Solution. $(2a^m + 3b^{-n}) \times (2a^m - 3b^{-n})$ is the square of $2a^m$, or $4a^{2m}$, minus the square of $3b^{-n}$, or $9b^{-2n}$. Therefore, $(2a^m + 3b^{-r})(2a^m - 3b^{-n}) = 4a^{2m} - 9b^{-2n}$.

Similarly, $(8p^4 + 11z^{\frac{1}{2}})(8p^4 - 11z^{\frac{1}{2}}) = 64p^8 - 121z$.

Exercise 18.

Write by inspection the product of the following:

1. $(2x + 3y)(2x - 3y)$; $(x + 2y)(x - 2y)$; $(5 + 3x)(5 - 3x)$; $(5x + 11)(5x - 11)$.

2. $(2x + 1)(2x - 1)$; $(2x + 5)(2x - 5)$; $(5xy + 3)(5xy - 3)$; $(c + a)(c - a)$.

3. $(c^2 - a^2)(c^2 - a^2)$; $(mn + 1)(mn - 1)$; $(ay^2 + b)(ay^2 - b)$; $(a^2x^2 + 1)(a^2x^2 - 1)$.

4. $(x^4 + y^4)(x^4 - y^4)$; $(1 - pq)(1 + pq)$; $(m - n)(m + n)$; $(a^m + a^n)(a^m - a^n)$.

5. $(5xy^{-1} - 4y^2)(5xy^{-1} + 4y^2)$; $(5x^2 + 3y^5)(5x^2 - 3y^5)$; $(x^3 - 3x)(x^3 + 3x)$.

6. $(2ax + by)(2ax - by)$; $(m^{-p} + n^{-q})(m^{-p} - n^{-q})$; $(10a^{-m} - 13b^{-n})(10a^{-m} + 13b^{-n})$.

7. $(m^{\frac{1}{2}} + n^{\frac{1}{2}})(m^{\frac{1}{2}} - n^{\frac{1}{2}})$; $(4a^{\frac{1}{2}} - 20x^{10})(4a^{\frac{1}{2}} + 20x^{10})$; $(a^{\frac{1}{3}} - b^{-\frac{2}{3}})(a^{\frac{1}{3}} + b^{-\frac{2}{3}})$.

8. $(11x^{\frac{1}{2}} + 30y^{\frac{1}{2}})(11x^{\frac{1}{2}} - 30y^{\frac{1}{2}})$; $(15a^2b^3 - 16a^{\frac{1}{2}}b^{\frac{3}{4}})(15a^2b^3 + 16a^{\frac{1}{2}}b^{\frac{3}{4}})$.

9. $(\frac{1}{2}ab^{-2} + \frac{2}{3}b^{-1}x^{-1})(\frac{1}{2}ab^{-2} - \frac{2}{3}b^{-1}x^{-1})$; $(a + b)(a - b)(a^2 + b^2)$.

10. $(ab+1)(ab-1)(a^2b^2+1)$; $(2a^m+4a^n)(2a^m-4a^n)(4a^{2m}+16a^{2n})$.

11. $(5a^3+6b^2)(5a^3-6b^2)(25a^6+36b^4)$; $(a^{-3}+a^2b^3)(a^{-3}-a^2b^3)(a^{-6}+a^4b^6)$.

12. $(x^{-\frac{2}{3}}+x^{-1}y)\ (x^{-\frac{2}{3}}-x^{-1}y)\ (x^{-\frac{4}{3}}+x^{-2}y^2)$; $(\frac{5}{4}c^{-m}+\frac{4}{5}b^n)(\frac{5}{4}c^{-m}-\frac{4}{5}b^n)(\frac{25}{16}c^{-2m}+\frac{16}{25}b^{2n})$.

Queries. In finding the product of monomials, why add exponents of like factors? What is the product of a^5 and a^3? Prove it. Why is the product of an even number of negative factors positive? How prove (1) and (2) Art. 25?

Miscellaneous Exercise 19.

Multiply:

1. $2a^{2n}-a^n+3$ by $2a^{2n}+a^n-3$; $5+2x^{2a}+3x^a$ by $4x^a-3x^{2a}$.

2. $a^x+2a^{2x}-3$ by $5-\frac{1}{3}a^x+2a^{2x}$; $\frac{1}{3}x^{\frac{1}{3}}-5+8x^{\frac{2}{3}}$ by $\frac{1}{4}x^{\frac{1}{3}}+\frac{1}{8}x^{-1}$.

3. $\frac{3}{4}a^{\frac{3}{5}}-a-a^{\frac{2}{5}}$ by $\frac{2}{3}a^{\frac{1}{5}}+a^{-1}-6a^{-\frac{1}{5}}$; $2a^{m-p}b^{-3p}+a^{-2p}b^p$ by $3a^mb^{4p}-a^{3p}b^{-2p}$.

4. $5x^ay^b-3x^{-a}y^{-b}$ by $4x^ay^b+5x^{2a}y^{2b}$; $a^{\frac{3}{2}n}+a^{-\frac{3}{2}n}$ by $a^{\frac{1}{2}n}+a^{-\frac{1}{2}n}$.

5. $.3a^4-.02a^3b+1.3a^2b^2+.5ab^3-1.2b^4$ by $.3a^2-.5ab-.6b^2$.

6. $1-2x^{\frac{1}{3}}-2x^{\frac{1}{6}}$ by $1-x^{\frac{1}{6}}$; $a^{\frac{3}{2}}-8a^{-\frac{3}{2}}+4a^{-\frac{1}{2}}-a^{\frac{1}{2}}$ by $\frac{1}{4}a^{-\frac{5}{2}}+a+\frac{1}{4}a^{-1}$.

7. $2x^{\frac{1}{3}}-x^{\frac{5}{3}}-3x^{-1}$ by $2x^2-3x^{-\frac{2}{3}}-x^{-2}$; a^n-1+a^{-n} by $a^{\frac{n}{2}}+a^{-\frac{n}{2}}$.

8. $x^{2n+1} - x^{n+1} - x^n + x^{n-1}$ by $x^{n+2} - x^2 - x + 1$; $x^n + 3x^{n-2} - 2x^{n-1}$ by $2x^{n+1} + x^{n+2} - 3x^n$.

9. $2x^{4n-1} - 3x^{3n} + x^{2n+1} - x^{n+2}$ by $x^{4n-1} - .5x^{3n} - .2x^{2n+1} + x^{n-2}$.

10. $3x^m - 2x^{m+1} - 5x^{m+2} + x^{m+3}$ by $3x^{n-3} + 2x^{n-2} - 5x^{n-1} - x^n$.

11. $x^{a+1} - 3x^{a+2} + x^{a-3} - 2x^{a+4}$ by $2x^{c-a} + 3x^{2c-a} - 4x^{3c-a}$.

12. $5x^{a-3}y^{c+3} - 2x^{a-1}y^{c+1} - x^{a-2}y^{c+2}$ by $3x^{a+4}y^{c-1} + 4x^{a+5}y^{c-2} - x^{a+3}y^c$.

13. $m^{p+1} - 3m^pn + m^{p-1}n^2 - m^{p-2}n^3$ by $m^{r-1} - 3m^rn + m^{r+2}n^2$.

14. $2x^{a+5}y^{c-1} + 3x^{a+4}y^{2c-1} - x^{a+3}y^{3c-1} + 4x^{a+2}y^{4c-1}$ by $2x^{a+2}y^{2-4c} - 3x^{a+1}y^{2-3c} + x^ay^{2-2c} + 4x^{a-1}y^{2-c}$.

15. $x^{a+4}y^{1-c} + x^{a+3}y^{2-c} - 2x^{a+c}y^{3-c} - 4x^{a+1}y^{4-c} + x^ay^{5-c}$ by $2x^{5-a}y^c - 2x^{4-a}y^{c+1} + 6x^{3-a}y^{c+2} - 2x^{2-a}y^{c+3} + 4x^{1-a}y^{c+4}$.

16. $(y^a + x^{-m})(y^a - x^{-m})$; $(\frac{3}{4}x^ay^{-b} - \frac{5}{7}x^{-a}y^b)(\frac{3}{4}x^ay^{-b} + \frac{5}{7}x^{-a}y^b)$.

17. $(x^n + y^m)(x^n - y^m)$; $(x^{\frac{1}{2}} - 5)(x^{\frac{1}{2}} + 4)$; $(7x - 3y^{-1})(7x + 3y^{-1})$.

18. $(4x - 5x^{-1})(4x + 3x^{-1})$; $(\frac{5}{4}c^{\frac{3}{2}}b^{-\frac{3}{2}} - \frac{9}{11}a^{\frac{n}{2}}b^{\frac{m}{2}})(\frac{5}{4}c^{\frac{3}{2}}b^{-\frac{3}{2}} + \frac{9}{11}a^{\frac{n}{2}}b^{\frac{m}{2}})$; $(a^n + 7 + 3a^{-n})(a^n + 7 - 3a^{-n})$.

CHAPTER V.

INVOLUTION.

27. Involution is the operation of raising an expression to any required power.

Involution may always be effected by taking the expression, as a factor, a number of times equal to the exponent of the required power.

It is evident from the law of signs that *even powers of any number are positive; and that odd powers of a number have the same sign as the number itself.* Thus,

$$(-m^3n)^2 = (-m^3n) \times (-m^3n)$$
$$= +m^{3+3}n^{1+1} \qquad = +m^6n^2.$$

$$(-m^4n^3)^3 = (-m^4n^3) \times (-m^4n^3) \times (-m^4n^3)$$
$$= -m^{4+4+4}n^{3+3+3} \qquad = -m^{12}n^9.$$

$$(-3m^3n)^4 = (-3m^3n) \times (-3m^3n) \times (-3m^3n) \times (-3m^3n)$$
$$= +3^{1+1+1+1}m^{3+3+3+3}n^{1+1+1+1} = +81m^{12}n^4.$$

$$(a^mb^c)^n = a^mb^c \times a^mb^c \times a^mb^c \times \ldots \text{ to } n \text{ factors}$$
$$= (a^m \times a^m \times a^m \times \ldots \text{ to } n \text{ factors}) \times (b^c \times b^c \times b^c \times \ldots \text{ to } n \text{ factors})$$
$$= (a^{m+m+m+\ldots \text{ to } n \text{ terms}}) \times (b^{c+c+c+\ldots \text{ to } n \text{ terms}})$$
$$= a^{m \times n} \times b^{c \times n}$$

$= a^{mn}b^{cn}$, where c, m, and n are *positive integers*, a and b may be integral or fractional, positive or negative.

Similarly, $(a^mb^cd^k \ldots p^r)^n = a^{mn}b^{cn}d^{kn} \ldots p^{rn}$. Hence, in general,

To Raise a Monomial to any Power. *Multiply the exponent of each factor by the exponent of the required power, and take the product of the resulting factors. Give to every even power the positive sign, and to every odd power the sign of the monomial itself.*

Notes: 1. Since, $-a^m = -1 \times a^m$, the nth power of $-a^m = (-1 \times a^m)^n = (-1)^n \times a^{mn}$. Or we may write $\pm a^{mn}$, for the nth power of $-a^m$, where the positive or negative sign is to be prefixed, depending upon the value of n whether an *even* or *odd* integer, m being positive and integral.

2. Any power of a fraction is found by taking the required power of each of its terms. Thus, $\left(\frac{m}{n}\right)^3 = \frac{m^3}{n^3}$; $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$.

Illustrations.

$$(-2\,x\,y^2\,z^3)^4 = +\,2^{1\times 4}\,x^{1\times 4}\,y^{2\times 4}\,z^{3\times 4} = +\,16\,x^4\,y^8\,z^{12}.$$
$$(-3\,x^2\,y^5)^3 = -\,3^{1\times 3}\,x^{2\times 3}\,y^{5\times 3} = -\,27\,x^6\,y^{15}.$$
$$(4\,m^a\,n^c)^x = +\,4^{1\times x}\,m^{a\times x}\,n^{c\times x} = +\,4^x\,m^{ax}\,n^{cx}.$$

Exercise 20.

Write the results of the following:

1. $(4\,a^2\,b^4)^2$; $(3\,a^6\,b^8)^3$; $(2\,x^4\,y^6)^5$; $(.2\,a^4 b^2\,c^6)^5$; $(.1\;a^n b^n)^2$; $(3\;a\;b^3)^3$; $(a^3\,c^2)^2$.

2. $(7\;a^3\;b^2)^2$; $(11\,a\,b^2\,c^3\,d^{10})^2$; $(-3\,c\,x^3\,y^4\,z^5)^3$; $(3\;a^5\,b^2\,y^3)^3$; $(5\;a\;b\;x^2\,y^5\,z^{10})^2$.

3. $(-2\,a\,b\,c^2 x^3\,y^4)^8$; $(-\;a\,b\,c\,d\,x)^{10}$; $(-a^3 b^2 c)^3$; $(-\,c^2\,b^2)^4$; $(3\;a\;b^3\,c^4)^6$; $(-\,2\;a\,b^2)^3$.

4. $(1\times a^4\,b^3\,c^2\,n)^{11}$; $(-2\,a^{2n}\,b^m)^5$; $(-3\,x\,y\,z)^3$; $(x^n\,y^m\,z^{5mn})^m$; $(a^4\,b^3\,c\,d^2)^n$.

5. $(-\,2\,a^r\,b^{6n})^3$; $(m^n)^n$; $(a^a)^a$; $(a\,b^{-1}\,c^{-2})^5$; $(m^n\,n^{-m})^{-mn}$; $(-\,2)^3$; $(-\,a)^{2n}$; $(-1)^{2n}$.

6. $n\,(2\,a\,b^{-2}\,c\,n^{-\frac{1}{8}})^4$; $n\,(n^2\,m^{3mn})^n$; $m\,(m^m\,a^{-1})^m$; $m\,n\,(3\,m^3\,n^4\,x^5\,y^{10})^2$.

7. $2\,a\,(-3\,m\,n^3x^4)^3$; $m\,(-3\,a^{10}b^8c^6m^4)^4$; $a^{10}\,(3\,a^{-2}\,b^{-1})^4$; $a\,(a^{a-1})^a$.

8. $(2\,x^{\frac{1}{4}}\,y^{\frac{1}{8}}\,z^{\frac{1}{16}})^4$; $(-k^n\,x^r\,y^{2n}\,z^{2m})^m$; $(-3\,a^{\frac{1}{2}k}\,b^{-3k}\,c)^{2k}$; $(-3\,a^{-4}\,c^{\frac{2}{3}}\,x^{\frac{4}{5}}\,y^{\frac{2}{15}})^5$.

Affect the following with the exponent 7; that is, raise each to the 7th power.

9. $-a^2b^3c^8d^4x^{\frac{1}{7}}$; $-x^my^n$; $a^{-m}b^ck^2$; $(-2^{\frac{1}{7}}a^3b^2)^3$; $(-x^{-3}y^n)^4$.

10. $(-x^ny^m)^6$; $(a^{\frac{1}{2}}b^{\frac{3}{2}})^6$; $(4^{\frac{1}{7}}\,a\,b\,x^n)^n$; $[(-x^{mn})^n]^m$; $(-k^km^m)^2$.

Write the nth powers of:

11. $m\,(a-3\,d)^p\,(x-y)^q$; $(a-3\,d)^{\frac{1}{2}n}\,(x-y)^2$; $3\,(a-b+c+d)(a-x)^m$.

12. $a\,b\,c\,(a-b^n)^m\,(x+y+z^m)^n$; $a^n\,(x-y+z)^{2n^2}\,(x-y^m)^{3n^3}$.

28. It may be shown by actual multiplication that:

$$
\begin{aligned}
(a+b)^2 &= a^2+b^2+2ab;\\
(a-b)^2 &= a^2+b^2-2ab;\\
(a+b+c)^2 &= a^2+b^2+c^2+2ab+2ac+2bc;\\
(a-b-c)^2 &= a^2+b^2+c^2-2ab-2ac+2bc;\\
(a+b+c+d)^2 &= a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd;\\
&\text{etc.} \qquad\qquad \text{etc.} \qquad\qquad \text{etc.}
\end{aligned}
$$

In each of the above products, observe that the square consists of:

I. *The sum of the squares of the several terms of the given expression.*

II. *Twice the algebraic product of the several terms taken two and two.*

These laws hold good for the square of all expressions, whatever be the number of terms. Hence, in general,

To Square any Polynomial. *Add together the squares of the several terms and twice the algebraic product of every two terms.*

EXAMPLE 1. Square $3a^3 - 4x^5$.

Solution. The squares of the terms are $9a^6$ and $16x^{10}$. Twice the algebraic product of the terms is $-24a^3x^5$.

Therefore, $(3a^3 - 4x^5)^2 = 9a^6 + 16x^{10} - 24a^3x^5$.

EXAMPLE 2. Square $2x^3 - 3x^2 - 1$.

Solution. The squares of the terms are $4x^6$, $9x^4$, and 1. Twice the algebraic product of the first term and each of the other two terms gives the products $-12x^5$ and $-4x^3$. Twice the product of the second and third terms is $6x^2$.

Therefore, $(2x^3 - 3x^2 - 1)^2 = 4x^9 + 9x^4 + 1 - 12x^5 - 4x^3 + 6x^2$.

Illustrations.

$$\begin{aligned}(2a^m - 3x^{-n})^2 &= (2a^m)^2 + (-3x^{-n})^2 + 2(2a^m) \times (-3x^{-n})\\ &= 4a^{2m} + 9x^{-2n} - 12a^m x^{-n}.\end{aligned}$$

$$\begin{aligned}(x^{-2}y^n - \tfrac{1}{2}x^n y^{-2} + \tfrac{2}{3}y^3 - \tfrac{1}{3}y)^2 &= (x^{-2}y^n)^2 + (-\tfrac{1}{2}x^n y^{-2})^2 + (\tfrac{2}{3}y^3)^2 + (-\tfrac{1}{3}y)^2\\ &\quad + 2(x^{-2}y^n) \times (-\tfrac{1}{2}x^n y^{-2}) + 2(x^{-2}y^n)\\ &\quad \times (\tfrac{2}{3}y^3) + 2(x^{-2}y^n) \times (-\tfrac{1}{3}y) + 2(-\tfrac{1}{2}x^n y^{-2})\\ &\quad \times (\tfrac{2}{3}y^3) + 2(-\tfrac{1}{2}x^n y^{-2}) \times (-\tfrac{1}{3}y) + 2(\tfrac{2}{3}y^3)\\ &\quad \times (-\tfrac{1}{3}y)\\ &= x^{-4}y^{2n} + \tfrac{1}{4}x^{2n}y^{-4} + \tfrac{4}{9}y^6 + \tfrac{1}{9}y^2 - x^{n-2}y^{n-2}\\ &\quad + \tfrac{4}{3}x^{-2}y^{n+3} - \tfrac{2}{3}x^{-2}y^{n+1} - \tfrac{2}{3}x^n y + \tfrac{1}{3}x^n y^{-1}\\ &\quad - \tfrac{4}{9}y^4.\end{aligned}$$

Exercise 21.

Square, by inspection, the following:

1. $x+2$; $m+5$; $n+7$; $a-10$; $2x+3y$; $a+3b$; $a-3b$; $2x-3y$.

2. $x+5y$; $3x-5y$; $2a+ab$; $5x-3xy$; $5abc-c$; $xy^{-1}-2y^2$; a^m+3b^{-n}.

3. $2x+3a^{\frac{1}{2}}$; $xy+x^{-\frac{2}{3}}$; $3a^{-2}+5a^5b^{-3}$; $1-x$; $1-cy$; $m-1$; ab^2-1; $\frac{7}{8}a^n-.05$.

4. $\frac{1}{2}ab^{-2}+\frac{2}{3}b^{-1}x^{-1}$; $p^nq^m-r^x$; $\frac{3}{4}a^{-n}-\frac{2}{3}b^{-m}$; $x^{-\frac{3}{4}}y^{-\frac{2}{3}}+\frac{1}{4}$; $.0002x^m+.005y^n$.

5. $\frac{2}{5}m^2n^3p^4-\frac{5}{2}mn^x$; $xy+yz+xz$; $2x^2+3x-1$; x^2-2x+1; x^2+2x-4.

6. $2x^2-x+3$; x^3-5x-2; $x^2-2xy+y^2$; $4n^2+m^2n-n^4$; x^4-3x+2.

7. $xy-2n+1$; $m-n-p-q$; x^3-2x^2+2x-3; $1+x+x^2+x^3$; $x+3y+2a-b$.

8. $2x^3-3x^2-x+3$; $x-2y-3z+2n$; $m^n+n^m+p^n-q^m$; $\frac{1}{3}a-3b-\frac{3}{2}$.

9. $\frac{1}{2}a-2b+\frac{1}{4}c$; $x^n-y^m+\frac{1}{2}a-\frac{1}{3}b$; $\frac{2}{3}x^2-x+\frac{3}{2}$; $\frac{3}{2}x^4-\frac{2}{3}x^3+\frac{1}{2}$.

10. $1+\frac{1}{2}x-\frac{1}{3}x$; $\frac{1}{2}x^2-\frac{1}{3}x-\frac{1}{5}$; $\frac{1}{4}a^n-\frac{1}{8}a^m+\frac{1}{3}xy$; $2x^{\frac{2}{3}}+5x^{\frac{1}{2}}+7$.

11. $3x^{\frac{1}{2}}-2x^{\frac{1}{3}}+\frac{1}{3}x^{\frac{1}{4}}-x^{-\frac{1}{6}}$; $m^{\frac{1}{2}n}+x^{\frac{1}{2}a}y^{-\frac{3}{2}b}-\frac{3}{2}z^{3c}-3$; $2^{\frac{1}{2}}-3^{\frac{1}{2}}$.

29. Any Power of a Binomial. It may be shown by actual multiplication that:

$$(a + b)^3 = a^3 + 3\,a^2 b + 3\,a\,b^2 + b^3;$$
$$(a - b)^3 = a^3 - 3\,a^2 b + 3\,a\,b^2 - b^3;$$
$$(a + b)^4 = a^4 + 4\,a^3 b + 6\,a^2 b^2 + 4\,a\,b^3 + b^4;$$
$$(a - b)^4 = a^4 - 4\,a^3 b + 6\,a^2 b^2 - 4\,a\,b^3 + b^4;$$
$$(a + b)^5 = a^5 + 5\,a^4 b + 10\,a^3 b^2 + 10\,a^2 b^3 + 5\,a\,b^4 + b^5;$$
$$(a - b)^5 = a^5 - 5\,a^4 b + 10\,a^3 b^2 - 10\,a^2 b^3 + 5\,a\,b^4 - b^5;$$

and so on.

In each of the above products we observe the following laws:

I. *The number of terms is one more than the exponent of the binomial.*

II. *If both terms of the binomial are positive, all the terms are positive.*

III. *If the second term of the binomial is negative, the odd terms, in the product, are positive, and the even terms negative.*

IV. *The first and the last terms of the product are respectively the first and the last terms of the binomial raised to the power to which the binomial is to be raised.*

V. *The exponent of the first term of the binomial, in the second term of the product, is one less than the exponent of the binomial, and in each succeeding term it decreases by one.*

The exponent of the second term of the binomial, in the second term of the product, is one, and in each succeeding term it increases by one.

Thus, omitting coefficients,

$$(a + b)^6 = a^6 + a^5 b + a^4 b^2 + a^3 b^3 + a^2 b^4 + a\,b^5 + b^6.$$

VI. *The coefficient of the first and the last term is one, that of the second term is the exponent of the binomial.*

The coefficient of any term, multiplied by the exponent of the first term of the binomial in that term, and divided by the number of the term, will be the coefficient of the next term.

Notes: 1. The sum of the exponents in any term of the expansion is the same, and is equal to the exponent of the binomial.

2. The coefficients of terms equally distant from the first term and the last term of the expansion are equal. Thus, we may write out the coefficients of the last half of the expansion from the first half.

If one or both of the terms of the binomial have more than one literal factor, or a coefficient or exponent other than 1, or if either of them is numerical, enclose it in parentheses before applying the principles. Thus,

EXAMPLE 1. Expand $(2x^3 - 5a^2x)^4$.

Process.

$$
\begin{aligned}
(2x^3-5a^2x)^4 &= [(2x^3)-(5a^2x)]^4 \\
&= (2x^3)^4-4(2x^3)^3(5a^2x)+6(2x^3)^2(5a^2x)^2-4(2x^3)(5a^2x)^3 \\
&\quad +(5a^2x)^4 \\
&= 2^4x^{12}-4\times 2^3x^9\times 5a^2x+6\times 2^2x^6\times 5^2a^4x^2-4\times 2x^3 \\
&\quad \times 5^3a^6x^3+5^4a^8x^4 \\
&= 16x^{12}-4\times 8x^9\times 5a^2x+6\times 4x^6\times 25a^4x^2-4\times 2x^3 \\
&\quad \times 125a^6x^3+625a^8x^4 \\
&= 16x^{12}-160a^2x^{10}+600a^4x^8-1000a^6x^6+625a^8x^4
\end{aligned}
$$

Explanation. In the expansion the odd terms will be positive, and the even terms negative. The first term is $(2x^3)^4$, and the fifth or last is $(5a^2x)^4$. The exponent of $(2x^3)$ is 4, and in each succeeding term it decreases by 1. The exponent of $(5a^2x)$ is 1, and in each succeeding term it increases by 1. The coefficient of the second term is 4. For the second term we take the product of 4, $(2x^3)^3$, and $(5a^2x)$. To find the coefficient of the third term, we multiply the coefficient of the second term 4 by 3 (the exponent of $(2x^3)$ in that term), and divide the product by 2 (the number of the term), and have 6. Hence, the third term is $6(2x^3)^2(5a^2x)^2$. The coefficient of the fourth term is found by multiplying 6 (the coefficient of the third term) by 2 (the exponent of $(2x^3)$ in the third term), and

dividing the product by 3 (the number of the term). Hence, the fourth term is $4(2x^3)(5a^2x)^3$. Performing operations indicated, we have the required result.

EXAMPLE 2. Raise $1-\frac{2}{3}x^n$ to the fifth power.

Process.

$$(1-\tfrac{2}{3}x^n)^5 = (1)^5-5(1)^4(\tfrac{2}{3}x^n)+10(1)^3(\tfrac{2}{3}x^n)^2-10(1)^2(\tfrac{2}{3}x^n)^3+5(1)(\tfrac{2}{3}x^n)^4 - (\tfrac{2}{3}x^n)^5$$
$$= 1^5-5\times 1^4\times\frac{2}{3}x^n+10\times 1^3\times\frac{2^2}{3^2}x^{2n}-10\times 1^2\times\frac{2^3}{3^3}x^{3n}+5\times 1 \times\frac{2^4}{3^4}x^{4n}-\frac{2^5}{3^5}x^{5n}$$
$$= 1-\tfrac{10}{3}x^n+\tfrac{40}{9}x^{2n}-\tfrac{80}{27}x^{3n}+\tfrac{80}{81}x^{4n}-\tfrac{32}{243}x^{5n}.$$

Exercise 22.

Expand and simplify the following expressions:

1. $(a-b)^7$; $(a+x)^6$; $(a^2-ac)^4$; $(a^3-4)^4$; $(2+a)^4$; $(a-1)^5$; $(1-a)^5$; $(2a-3b)^4$.

2. $(x^{\frac{1}{2}}-3)^4$; $(ax-3x^2)^5$; $(x-3)^5$; $(2a^2x+3b^2y^2)^3$; $(2ax+3by)^4$.

3. $(a+2)^6$; $(a-2)^6$; $(2-\frac{1}{3}a)^4$; $(\frac{1}{2}a-3b)^4$; $(\frac{1}{2}a+\frac{1}{3}b)^4$; $(a+b)^{10}$.

4. $(a^{\frac{1}{2}}-2-a^{-\frac{1}{2}})^2$; $[(x+y)^2+(x-y)^2]^2$; $(1+a+a^2)^2-(1-a+2a^2)^2$.

5. $(a+2b)^4-(a-2b)^4$; $(3-2a+a^2)^2-(2-a)^4$; $(3^{\frac{1}{2}}+5^{\frac{1}{2}})^2-(2^{\frac{1}{2}}-3^{\frac{1}{2}})^2$.

Queries. How prove $(-m)^n = \pm m^n$, according to the value of n, whether an even or odd integer? How prove the method for squaring any polynomial? How prove the laws for raising a binomial to any power?

CHAPTER VI.

ALGEBRAIC DIVISION.

30. **Division** is the inverse of multiplication, and is the operation of finding the other factor, when a product and one of its factors are given. The product is now called the **Dividend**, the given factor is the **Divisor**, and the required factor is the **Quotient**. Thus,

since $a^2 \times a^3 = a^5$, $\therefore$ $a^5 \div a^3 = a^2$;
since $a^{-2} \times a^5 = a^3$, $\therefore$ $a^3 \div a^5 = a^{-2}$;
since $a^5 \times a^{-4} = a$, $\therefore$ $a \div a^{-4} = a^5$;
since $a^{m-n} \times a^n = a^m$, $\therefore$ $a^m \div a^n = a^{m-n}$;
since $a^{m+n} \times a^{-n} = a^m$, $\therefore$ $a^m \div a^{-n} = a^{m+n}$;
since $3a^3b^4 \times 2a^{-2}b = 6ab^5$, $\therefore$ $6ab^5 \div 2a^{-2}b = 3a^3b^4$;
since $9a^{-3}b^2 \times 3a^4b^5 = 27ab^7$, $\therefore$ $27ab^7 \div 3a^4b^5 = 9a^{-3}b^2$;
since $5a^{\frac{3}{4}}b^{-\frac{1}{3}} \times 4a^{-\frac{1}{4}}b^{\frac{2}{3}} = 20a^{\frac{1}{2}}b^{\frac{1}{3}}$, $\therefore$ $20a^{\frac{1}{2}}b^{\frac{1}{3}} \div 4a^{-\frac{1}{4}}b^{\frac{2}{3}} = 5a^{\frac{3}{4}}b^{-\frac{1}{3}}$;

etc. Hence, in general,

To Divide a Monomial by a Monomial. *To the quotient of the numerical coefficients annex the literal factors, each taken with an exponent obtained by subtracting its exponent in the divisor from its exponent in the dividend.*

Illustrations.

$$a^3b^4c^5m^2 \div a^2b^3c^2m = a^{3-2}b^{4-3}c^{5-2}m^{2-1} = abc^3m.$$

$$63a^{-2}b^2c^5 \div 7a^{-3}bc^4 = 9a^{-2+3}b^{2-1}c^{5-4} = 9abc.$$

$$15a^2b^2 \div 6bc = \frac{5a^2b^{2-1}}{2c} = \frac{5a^2b}{2c} \text{ (Art. 2).}$$

Exercise 23.

Divide:

1. $3a^3b^2$ by ab; $15a^4b^3$ by $3a^3b^2$; $20a^2b^3c^2$ by $5ab^2c^3$; $3m^{\frac{3}{4}}$ by $5m^{\frac{1}{3}}$.

2. n^{-5} by n^{-10}; a^n by a^{n-3}; $a^2b^{-3}c^n$ by $a^3b^{-2}c^2$; a^{m+n} by a^{m-n}; 2^{x+y} by 2^{y-x}.

3. $15a^{-\frac{2}{3}}b^{-\frac{1}{2}}x^2$ by $9a^{-2}b^{-1}x^3$; $\frac{6}{7}a^{\frac{1}{3}}b^{\frac{1}{2}}$ by $\frac{3}{2}a^{\frac{1}{2}}b^{\frac{1}{3}}$; $21a^nm^2x^c$ by $7amx^b$.

4. $24a^np^m$ by $3a^mp^n$; $36a^mm^2y^cn$ by $9amyn^b$; $x^{a+2}y^{a-2}$ by x^4y^2.

5. $(x-y)^5$ by $(x-y)^3$; $(a-c)^{b+3}$ by $(a-c)^{b-1}$; $\frac{3}{4}b^nt^mk^a$ by $\frac{2}{3}bt^3k^2$.

6. $(6a^3b^2c \times 15a^5b^6c^3)$ by $(5a^2b^7c^2 \times 2a^4c^3)$; a^{m^2} by a^{n^2}; $(2mn^a)^{2x}$ by $(2mn^a)^x$.

31. Only a positive number, $+a$, when multiplied by $+b$, can give the positive product $+ab$. Therefore, $+ab$ divided by $+b$ gives the quotient $+a$.

Thus, since $a \times b = +ab$, $\therefore +ab \div +b = +a$;
since $a \times -b = -ab$, $\therefore -ab \div -b = +a$;
since $-a \times b = -ab$, $\therefore -ab \div +b = -a$;
since $-a \times -b = +ab$, $\therefore +ab \div -b = -a$.

Hence, in general,

Law of Signs. *If the dividend and the divisor have the same sign, the quotient is positive. If they have opposite signs, the quotient is negative.*

EXAMPLE. Divide $12\,a^m$ by $-4\,a^n$.

Solution. Since there is a factor 4 in the divisor, there must be a factor 3 in the quotient, in order to give a product of 12 in the dividend. Since there are m factors of a in the dividend, and n in the divisor, there must be $m-n$ factors of a in the quotient, in order to give a product of a^m in the dividend. Hence, $12a^m \div -4\,a^n = -3\,a^{m-n}$, because only a negative number, $-3\,a^{m-n}$, when multiplied by $-4\,a^n$ can give the positive product, $12\,a^m$.

Illustrations.

$$-15\,a^5 m^6 b^3 \div 3\,a^2 m^4 b^2 = -5\,a^{5-2} b^{3-2} m^{6-4} = -5\,a^3 b\, m^2.$$

$$-5\,x^{10} y^3 z^6 \div -10\,x^8 y^5 z^3 = +\tfrac{1}{2}\,x^{10-8} y^{3-5} z^{6-3} = +\tfrac{1}{2}\,x^2 y^{-2} z^3.$$

$$7\,a^n (a-b)^3 (x+y)^m \div -4\,a(a-b)^2 (x+y)^n = -\tfrac{7}{4}\,a^{n-1}(a-b)\,(x+y)^{m-n}.$$

Exercise 24.

Divide:

1. $6x^2$ by $3x$; $-20\,a^5 b^6 c^7$ by $10\,abc$; $35\,a^{11}$ by $-7\,a^7$; $-7\,a^3 b\, c^2$ by $-7\,a^3 b\, c^2$.

2. $27\,a\,x^4$ by $-9\,x^4$; $-\frac{3}{4}\,a^6 b^6 c^6$ by $\frac{2}{3}\,a^4 b\, c^2$; $.5\,a^7 b^{10} c^{13}$ by $\frac{5}{3}\,a^6 b^{11} c^{14}$; $12\,x^{2n} y^{\frac{1}{2}}$ by $-\frac{2}{5}\,x^n y$.

3. $3\frac{1}{3}\,m^2 n^5 x^2$ by $-2\frac{1}{9}\,m^{-1} n^{-3} x^{-2}$; $-5\frac{2}{3}\,m^{-3} x^{-1} y^{10}$ by $1\frac{2}{15}\,a^2 m^3 x^{-4} y$; $3.2\frac{7}{8}\,a^{-5} x^3 y^5$ by $2.02\frac{3}{4}\,a^{-6} x\, y^2$.

4. $.0\frac{7}{16}\,a^3 m^2 y^3 z^4$ by $-.0\frac{1}{7}\,a^2 m\, y^2 z^3$; $-9.3\,m^{3a+2} x^{n-3} y^{\frac{2}{3}c}$ by $.3\,m^{3a+1} x^{n-4} y^{\frac{1}{3}m}$.

5. $.6\dot{6}\,x^a y^b$ by $-.\dot{1}\,x^c y^{-2b}$; $-\frac{7}{8}(a\quad b)^3 c^8$ by $.6\,(a-b)^2 c^{10}$; $-.3\,a^{\frac{1}{2}m} b^{\frac{2}{3}n}$ by $-.2\,a^m b^n$.

6. $-.375\, x^{\frac{3}{2}}\, y^{\frac{2}{3}}\, (x^{\frac{3}{2}} - y^{\frac{2}{3}})^{\frac{3}{4}}$ by $-\frac{3}{80}\, x^{\frac{1}{3}}\, y\, (x^{\frac{3}{2}} - y^{\frac{2}{3}})^{\frac{1}{3}}$; $8\, m^{.5}\, n^{-1}\, x^{-.02}\, y^{.7}$ by $9\, m^{-.5}\, n^{\;\;2}\, x^{.08}\, y^{-.3}$.

7. $-1.2\, a^{10}(x-y)^{7} z^{.9}$ by $.3\, a^{5}(x-y)^{5} z^{1.1}$; $m^{-a}\, n^{b}(x-y)^{c}(y-z)^{d}$ by $m^{-2a}\, n^{2b}(x-y)^{-c}(y-z)^{\frac{1}{2}d}$.

Simplify the following, that is, perform the indicated operations:

8. $(3\, a^2 b^2 c \times 12\, a^{-1} b^2 c^3) \div 6\, a^3 b^3 c^3$; $(32\, a^m b^n c^x \div 8\, a^n b^m) \times -.5\, a^{2n}\, b^{2m}\, c^{-2x}$.

9. $(a^{-2} b^4 \div 2\, a\, b) \times -2\, a^2 b^{-2} \times (-.6\, a^{\frac{1}{3}} b^{\frac{1}{2}} \div -.3\, a^{\frac{1}{2}} b^{\frac{1}{3}})$.

10. $(.3\, a^{-m}\, b^{-n}\, c^{-p} \div .03\, a^m b^n c^p) \div 1\frac{2}{3}\, a^{-3m}\, b^{-3n}\, c^{-3p}\, k$.

11. $(4\frac{2}{3}\, a^{-9}\, b\, d\, x^{-2} \div 1\frac{1}{6}\, a^{-7}\, b^{-3}\, d^4)$
$\times [6\, a^2 c^{-1} d^3\, x \div (84\, a^3\, b^3\, c \div 7\, a^4\, b^3\, c^2)]$.

12. $(x^{m\;\;n} \times a^{-\frac{1}{2}}\, b^{-\frac{3}{2}}\, x^{m-n}) \times (a^{\frac{1}{2}}\, b\, x^{3m}\, y^{\frac{1}{4}} \div b^{\frac{1}{2}}\, x^{2m}\, y^{-\frac{3}{4}})$.

13. $(-14\, a^{10}\, b^6\, c^{-3} \div -7\, a^5\, b^4\, c^{-4})$
$\div (28\, a^m b^n c^p \div -4\, a^{-m} b^{-n} c^{-p})$.

14. $(1.7\, a^{-\frac{2}{3}} b^{-\frac{1}{2}} c^{\frac{7}{2}} x^2 \div 1.1\, a^{-2} b^{-1} x^3) \times (a^m b^n c^{-5} \div a^{\frac{4}{3}} b^2 c^3)$.

32. Since $(a+b)\, m = a\, m + b\, m$, $\therefore$ $(a\, m + b\, m) \div m = a + b$.

Since $(a-b)\, m = a\, m - b\, m$, $\therefore$ $(a\, m - b\, m) \div m = a - b$.

Since $(x\, y - 2\, y^2 z - 3\, x^3 y^{-2}) \times -3\, x\, y^3 = -3\, x^2 y^4 + 6\, x\, y^5 z + 9\, x^4 y$,

$\therefore$ $(-3\, x^2 y^4 + 6\, x\, y^5 z + 9\, x^4 y) \div -3\, x\, y^3 = x\, y - 2\, y^2 z - 3\, x^3 y^{-2}$.

Hence, in general,

To Divide a Polynomial by a Monomial. *Divide each term of the dividend by the divisor, and add the results.*

Exercise 25.

Divide:

1. $2a^2 + 6a^3y - 8a^4y^2$ by $2a^2$; $21m^3n^3 - 7m^2n^2 - 14mn + 63$ by $7mn$.

2. $a^2bc - a^2b^3c^2 - a^3b^2c^3 + a^4b^4c^3$ by a^2bc; $4.2x^3 - 1.1x^2 + 28x$ by $.7x$.

3. $28a^3 + 9a^2 - 21a + 35$ by $7a$; $4a^3b^2 - 16a^4b^4 + 4a^2b^3$ by $-4a^3b^3$.

4. $6a^2b^2c^3 - 48a^2b^4c^2 + 36a^2b^2c^4 - 20abc^6$ by $4abc^2$.

5. $2.4m^2n^2 - .8m^4n^5 - 2.4mn^2 + 4m^2n^3$ by $.8mn$; $x^{\frac{3}{4}} - x^{\frac{3}{5}}y^{\frac{2}{3}}$ by $x^{\frac{1}{4}}$.

6. $-3a^2 + \frac{9}{2}ab - 6ac$ by $-1.5a$; $.5m^5n^2 - 3m^3n^4$ by $-1.5m^3n^2$.

7. $-72a^5c^2 - 48a^7c^{10} + 32a^2c^3$ by $16a^3c^3$; $3.6n^{\frac{7}{2}} - 4.8n^{\frac{3}{4}}$ by $4n^{\frac{1}{3}}$.

8. $11x^2y^3 + 3xn - 2.4y^2$ by $.3xy$; $.09m^{14} - 2.4m^3n + 4.8m^5$ by $.03m^4$.

9. $-a^m + 2a^{-m} - 3a^n$ by $-a^2$; $m^{n+1} - m^{n+2} + m^{n+3} - m^{n+4}$ by m^3.

10. $2.1ax^2y^m + 1.4a^3x^4y^n - 2.8a^5x^2y^p$ by $-.7axy^n$.

11. $a^mb^3 - a^{m+1}b^2 + a^{n-2}b$ by $-ab$; $-2a^5x^3 + 3.5a^4x^4$ by $2.3\frac{1}{3}a^3x$.

12. $2.25\,a^2x - .0625\,abx - .375\,acx$ by $.37\tfrac{1}{2}\,ax$; $11\,x^{\frac{5}{9}} - 33\,x^{\frac{4}{3}}$ by $11\,x^{\frac{2}{9}}$.

13. $72\,m^{\frac{5}{6}} - 60\,m^{\frac{1}{6}}n^{\frac{2}{3}} + 12\,m^{\frac{2}{3}}n^{\frac{1}{6}} - 6\,m^{\frac{1}{12}}n^{\frac{1}{6}}$ by $24\,m^{\frac{1}{6}}$.

14. $36\,(x-y)^5 - 27\,(x-y)^3 + 18\,(x-y)$ by $9\,(x-y)$.

15. $-12\,x^m y^n z^t - 30\,x^{m+2}y^{2n}z + 108\,x^{2m}yz^{t+2}$ by $-6\,x^m y^n z$.

16. $m^n(x-y)^a - m^a(x-y)^n$ by $m^n(x-y)^n$.

17. $(x+y)^a(x-y)^n + (x+y)^n(x-y)^a$ by $(x+y)^c(x-y)^b$.

18. $-2.5\,m^2 + 1.\dot{6}\,mn + 3.\dot{3}\,m$ by $-.8\dot{3}\,m$; $a^{.83\frac{1}{3}} - a^{.16\frac{2}{3}}b^2 + a^{.\dot{6}}$ by $a^{.1\dot{6}}$.

33. It may be shown by actual multiplication that:

$$(m+n+p)\ (x+y+z) = mx+my+mz+nx+ny+nz+px+py+pz.$$
$$\therefore (mx+my+mz+nx+ny+nz+px+py+pz) \div (x+y+z) = m+n+p.$$

The division is performed as follows:

Separate the dividend into the three parts $mx + my + mz$, $nx + ny + nz$, and $px + py + pz$. The first term of the quotient, m, is found by dividing mx, the first term of the dividend, by x, the first term of the divisor; multiplying the entire divisor by m will produce the *first part* of the dividend. The second term n of the quotient is found by dividing the first term of the second part of the dividend by the first term of the divisor; multiplying the entire divisor by n will produce the *second part* of the dividend. The third term p of the quotient is found by dividing the first term of the third part of the dividend by the first term of the divisor; multiplying the entire divisor by p will produce the *third part* of the dividend. The work is conveniently arranged as follows:

Process.

	Divisor.	Dividend.	Quotient.
		First part. Second part. Third part.	
	$x+y+z$	$\overbrace{mx+my+mz}+\overbrace{nx+ny+nz}+\overbrace{px+py+pz}$	$m+n+p$
Product of entire divisor and first term of quotient,		$mx+my+mz$	
First remainder, or second and third parts,		$nx+ny+nz+px+py+pz$	
Product of entire divisor and second term of quotient,		$nx+ny+nz$	
Second and last remainder, or third part,		$px+py+pz$	
Product of entire divisor and last term of quotient,		$px+py+pz$	

EXAMPLE 1. Divide $x^7 - 5x^5 + 7x^3 + 2x^2 - 6x - 2$ by $x^4 - 3x^2 + 2x + 1$.

Process.

	Divisor.	Dividend.	Quotient.
	$x^4 - 3x^2 + 2x + 1$	$x^7 - 5x^5 + 7x^3 + 2x^2 - 6x - 2$	$x^3 - 2x - 2$
x^3 times the entire divisor,		$x^7 - 3x^5 + 2x^4 + x^3$	
First remainder,		$-2x^5 - 2x^4 + 6x^3 + 2x^2 - 6x - 2$	
$-2x$ times the entire divisor,		$-2x^5 + 6x^3 - 4x^2 - 2x$	
Second and last remainder,		$-2x^4 + 6x^2 - 4x - 2$	
-2 times the entire divisor,		$-2x^4 + 6x^2 - 4x - 2$	

Explanation Dividing the first term of the dividend by the first term of the divisor, we have x^3, the first term of the quotient. Now as we are to find how many times $x^4 - 3x^2 + 2x + 1$ is contained in the dividend, and have found that it is contained x^3 times, we may take x^3 times the divisor out of the dividend, and then proceed to find how many times the divisor is contained in the remainder of the dividend. Dividing the first term of the remainder by the first term of the divisor, we have $-2x$, the second term of the quotient. Similarly, we find the third term of the quotient. Hence, the quotient is $x^3 - 2x - 2$.

Notes: 1. Algebraic division is strictly analogous to "long division" in Arithmetic. The arrangement of the terms corresponding to the order of succession of the thousands, hundreds, tens, units, etc., and the processes for both are exactly the same.

2. It is *convenient to arrange both dividend and divisor according to powers* of the same letter ascending or descending.

3. It may happen the division *cannot be exactly performed;* we then *algebraically* add to the quotient the fraction whose numerator is the remainder, and whose denominator is the divisor. Thus, if we divide $x^2 - 2xy - y^2$ by $x - y$, we shall obtain $x - y$ in the quotient, and there *will be a remainder* $-2y^2$. Hence, $(x^2 - 2xy - y^2) \div (x - y) = x - y - \dfrac{2y^2}{x-y}$.

EXAMPLE 2. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

Arranging according to the descending powers of a, we have:

Process. Divisor. Dividend. Quotient

$$a+b+c)\,a^3 \qquad -3abc+b^3+c^3\,(a^2-ab-ac+b^2+c^2-bc$$

a^2 times the divisor,	$a^3+a^2b+a^2c$
First remainder,	$-a^2b-a^2c \qquad -3abc+b^3+c^3$
$-ab$ times the divisor,	$-a^2b \qquad -ab^2 \qquad -abc$
Second remainder,	$-a^2c+ab^2 \qquad -2abc+b^3+c^3$
$-ac$ times the divisor,	$-a^2c \qquad -ac^2-abc$
Third remainder,	$ab^2+ac^2-abc+b^3+c^3$
b^2 times the divisor,	$ab^2 \qquad +b^3+b^2c$
Fourth remainder,	$ac^2-abc-b^2c+c^3$
c^2 times the divisor,	$ac^2 \qquad +bc^2+c^3$
Fifth and last remainder,	$-abc-b^2c-bc^2$
$-bc$ times the divisor,	$-abc-b^2c-bc^2$

To verify the work, multiply the quotient by the divisor.

EXAMPLE 3. Divide $\frac{1}{72}xy^2 + \frac{1}{4}x^3 + \frac{1}{12}y^3$ by $\frac{1}{3}y + \frac{1}{2}x$.

Process.

$$
\begin{array}{lr}
\frac{1}{2}x + \frac{1}{3}y\,\big)\,\frac{1}{4}x^3 \qquad\qquad + \frac{1}{72}xy^2 + \frac{1}{12}y^3\,\big(\,\frac{1}{2}x^2 - \frac{1}{3}xy + \frac{1}{4}y & \\
\text{Divisor} \times \frac{1}{2}x^2, & \underline{\frac{1}{4}x^3 + \frac{1}{6}x^2y \qquad\qquad\qquad\qquad} \\
\text{First remainder,} & -\frac{1}{6}x^2y + \frac{1}{72}xy^2 + \frac{1}{12}y^3 \\
\text{Divisor} \times -\frac{1}{3}xy, & \underline{-\frac{1}{6}x^2y - \frac{1}{9}xy^2 \qquad\qquad} \\
\text{Second and last remainder,} & \frac{1}{8}xy^2 + \frac{1}{12}y^3 \\
\text{Divisor} \times \frac{1}{4}y^2, & \underline{\frac{1}{8}xy^2 + \frac{1}{12}y^3.}
\end{array}
$$

Hence, in general,

To Divide a Polynomial by a Polynomial. *Divide the first term of the dividend by the first term of the divisor for the first term of the quotient; multiply the entire divisor by this term, and subtract the product from the dividend. Divide as before, and repeat the process until the work is completed.*

Exercise 26.

Divide:

1. $14x^4 + 45x^3y + 78x^2y^2 + 45xy^3 + 14y^4$ by $2x^2 + 5xy + 7y^2$.

2. $x^4 - 2x^3y + 2x^2y^2 - xy^3$ by $x - y$; $a^3 - 2ab^2 + b^3$ by $a - b$.

3. $y^5 - 5y^4 + 9y^3 - 6y^2 - y + 2$ by $y^2 - 3y + 2$; $y^6 - 1$ by $y - 1$.

4. $x^2 + xy + 2xz - 2z^2 + 7yz - 3z^2$ by $x - y + 3z$; $a^8 - b^8$ by $a - b$.

5. $2xy + 3by + 10bx + 15b^2$ by $y + 5b$; $a^6 + a^5b$ by $a + b$.

6. $.125x^3 - 2.25x^2y + 13.5xy^2 - 27y^2$ by $.5x - 3y$.

7. $y - 6y^3 - 2x + 54x^3 - 3x^2y$ by $2x - y$; $x^4 - y^4$ by $x + y$.

8. $x^2y^2 - x^2 - y^2 + 1$ by $xy + x + y + 1$; $4y^5 + 4y - y^3$ by $3y + 2y^2 + 2$.

9. $x^3 + y^3 - z^3 + 3xyz$ by $x + y - z$; $x - y$ by $x^{\frac{1}{4}} - y^{\frac{1}{4}}$.

10. $x^2y^2 + 2xy^2z - x^2z^2 + y^2z^2$ by $xy + xz + yz$; $x^3 - y^2$ by $x^{\frac{3}{5}} - y^{\frac{2}{5}}$.

11. $12x^4 - 26x^3y - 8x^2y^2 + 10xy^3 - 8y^4$ by $3x^3 - 2xy + y^2$.

12. $x^3 + y^3 + 3xy - 1$ by $x + y - 1$; $\frac{1}{27}x^3 - \frac{1}{12}x^2 + \frac{1}{16}x - \frac{1}{64}$ by $\frac{1}{3}x - \frac{1}{4}$.

13. $12x^4y^9 - 14x^5y^6 + 6x^6y^3 - y^7$ by $2x^2y^3 - y^3$; $x^2 - y^2$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

14. $a^5b - ab^5$ by $a^3 + b^3 + ab^2 + a^2b$; $x^4 + x^{-4} - x^2 - x^{-2}$ by $x - x^{-1}$.

15. $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$ by $x^3 + y^3$; $a^{\frac{3}{2}} - b^{\frac{3}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

16. $\frac{1}{3}a^3 + \frac{17}{6}a^2 - 5.25a + 2.25$ by $\frac{1}{3}a + 3$; $.75x^2y^3 + .048x^5$ by $.2x^2 + .5xy$.

17. $a^{\frac{5}{2}} - a^2 - 4a^{\frac{3}{2}} + 6a - 2a^{\frac{1}{2}}$ by $a^{\frac{3}{2}} - 4a^{\frac{1}{2}} + 2$; $x^5 - y^5$ by $x - y$.

18. $x^3 + y^3 + z^3 + 3x^2y + 3xy^2$ by $x + y + z$; $.5x^3 + x^2 + .375x + .75$ by $\frac{1}{2}x + 1$.

19. $x^3 + 8y^3 + z^3 - 6xyz$ by $x^2 + 4y^2 + z^2 - xz - 2xy - 2yz$.

20. $\frac{9}{16}x^4 - \frac{3}{4}x^3 - \frac{7}{4}x^2 + \frac{4}{3}x + \frac{16}{9}$ by $1.5x^2 - x - \frac{8}{3}$.

21. $x^8 - y^8$ by $x^3 + x^2y + xy^2 + y^3$; $x^9 - y^9$ by $x^2 + xy + y^2$.

22. $10a^4 - 27a^3b + 34a^2b^2 - 18ab^3 - 8b^4$ by $5a^2 - 6ab - 2b^2$.

23. $36x^2 + \frac{1}{9}y^2 + .25 - 4xy - 6x + \frac{1}{3}y$ by $6x - \frac{1}{3}y - .5$.

24. $a^{12} + 2a^6b^6 + b^{12}$ by $a^4 + 2a^2b^2 + b^4$; $a^6 - b^6$ by $a^3 - 2a^2b + 2ab^2 - b^3$.

25. $2x^{3n} - 6x^{2n}y^n + 6x^ny^{2n} - 2y^{3n}$ by $x^n - y^n$; $x^{3n} + y^{3n}$ by $x^n + y^n$.

26. $x^{2n} - y^{2m} + 2y^mz^t - z^{2t}$ by $x^n + y^m - z^t$; $3^{2x} - 2^{2x}$ by $3^x - 2^x$.

27. $\frac{8}{27}x^5 - 2\frac{43}{512}xy^4$ by $\frac{2}{3}x - .75y$; $x^{-\frac{2}{3}m} - 3x^{-\frac{1}{3}m}y^{-\frac{1}{5}n} + 2y^{-\frac{2}{5}n}$ by $x^{-\frac{1}{3}m} - y^{-\frac{1}{5}n}$.

28. $y^2x^{2m} + 2yzx^{m+n} + 2yrx^m + z^2x^{2n} + 2rx^nz + r^2$ by $yx^m + zx^n + r$.

29. $x^{-1} + 2x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$ by $x^{-\frac{1}{2}} + y^{-\frac{1}{2}}$; $x^4 + y^4$ by $x^2 + 2^{\frac{1}{2}}xy + y^2$.

30. $x^{-1} - y^{-\frac{4}{3}} + 2y^{-\frac{2}{3}}z^{-\frac{1}{4}} - z^{-\frac{1}{2}}$ by $x^{-\frac{1}{2}} + y^{-\frac{2}{3}} - z^{-\frac{1}{4}}$; $x^4 - 3y^4$ by $x - y$.

34. There are special methods for finding the quotient of binomials, *by inspection,* which are of importance on account of their frequent occurrence in algebraic operations. Thus,

It may be shown by actual division that:

$$\frac{a^3-b^3}{a-b} = a^2+ab+b^2; \qquad \frac{a^4-b^4}{a-b} = a^3+a^2b+ab^2+b^3;$$

$$\frac{a^5-b^5}{a-b} = a^4+a^3b+a^2b^2+ab^3+b^4; \quad \frac{a^6-b^6}{a-b} = a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5;$$

and so on. Hence, in general, it will be found that,

The difference of any two equal powers of two numbers is divisible by the difference of the numbers.

In each of the above quotients we observe the following laws:

I. *The number of terms is equal to the exponent of the powers.*

II. *The signs are all positive.*

III. *The exponent of* a *in the first term is one less than the exponent of* a *in the first term of the dividend, and in each succeeding term it decreases by one (in the last term its exponent is* 0, *or* a *disappears).*

The exponent of b *in the second term is one, and in each succeeding term it increases by one (in the last term its exponent is one less than the exponent of* b *in the dividend).*

IV. *The first term is found by dividing the first term of the dividend by the first term of the divisor.*

V. *To find each succeeding term, divide the preceding term by the first term of the divisor, and multiply the result by the second term of the divisor regardless of sign.*

EXAMPLE. Divide $1 - n^5$ by $1 - n$.

Solution. Dividing 1, the first term of the dividend, by 1, the first term of the divisor, we get 1 for the *first term* of the quotient. Now divide the first term of the quotient by the first term of the divisor, and multiply the result by n, the second term of the divisor (regardless of sign), for the *second term*, n, of the quotient. Dividing the second term of the quotient by the first term of the divisor, and multiplying the result by n, we have n^2 for the *third term* of the quotient. Similarly, we find n^3, and n^4 for the *fourth* and *fifth terms*, respectively. $\therefore (1 - n^5) \div (1 - n) = 1 + n + n^2 + n^3 + n^4$.

Exercise 27.

Divide by inspection:

1. $m^3 - n^3$ by $m - n$; $a^5 m^5 - b^5 n^5$ by $a m - b n$; $m^5 n^5 - 1$ by $m n - 1$.

2. $1 - m^6 n^6 x^6$ by $1 - m n x$; $(x y)^7 - (x z)^7$ by $x y - x z$; $1 - a^7 b^7 x^7$ by $1 - a b x$.

In order to apply this principle the terms of the dividend must be the *same powers* of the respective terms of the divisor. It is not necessary that the exponents of the terms of the divisor be 1, nor that they be the same, nor that the exponents of the terms of the dividend be the same. Thus,

EXAMPLE. Divide $x^{12} - y^{16}$ by $x^3 - y^4$.

Solution. Dividing x^{12} by x^3, we have x^9 for the *first term* in the quotient. Now divide x^9 by x^3 and multiply the result by y^4, for the *second term*, $x^6 y^4$, in the quotient. In like manner we find $x^3 y^8$, and y^{12} for the *third* and *fourth terms* of the quotient.

$$\therefore (x^{12} - y^{16}) \div (x^3 - y^4) = x^9 + x^6 y^4 + x^3 y^8 + y^{12}.$$

So in general $x^a - y^m$ divides $x^{na} - y^{nm}$ (n being any positive integer), since the dividend is the difference between the nth powers of the terms of the divisor.

3. $a^9 - b^6$ by $a^3 - b^2$; $x^{21} - y^{21}$ by $x^7 - y^7$; $x^{18} - y^{12}$ by $x^3 - y^2$.

4. $a^{15} - b^{30}$ by $a^3 - b^6$; $x^{21n} - y^{35n}$ by $x^{3n} - y^{5n}$; $2^{10n} - x^{5m}$ by $2^{2n} - x^m$.

We may easily apply these principles to examples containing coefficients as well as exponents; also to those involving fractional or negative exponents. Thus,

EXAMPLE. Divide $81a^{12} - 16b^{24}$ by $3a^3 - 2b^6$.

Solution. Dividing $81a^{12}$ by $3a^3$, we have $27a^9$ for the *first term* of the quotient. Now divide $27a^9$ by $3a^3$ and multiply the result by $2b^6$, for the *second term*, $18a^6b^6$, in the quotient. Similarly, we find $12a^3b^{12}$, and $8b^{18}$ for the *third* and *fourth terms* in the quotient.

$\therefore (81a^{12} - 16b^{24}) \div (3a^3 - 2b^6) = 27a^9 + 18a^6b^6 + 12a^3b^{12} + 8b^{18}$.

If a and b are coefficients, $a^n x^{np} - b^n y^{nm}$ is divisible by $ax^p - by^m$, since the dividend is the difference between the nth powers of ax^p and by^m. In general, $x^{-\frac{a}{m}} - y^{-\frac{s}{r}}$ divides $x^{-\frac{na}{m}} - y^{-\frac{ns}{r}}$ (n being *any positive* integer), since the latter is the difference between the nth powers of $x^{-\frac{a}{m}}$ and $y^{-\frac{s}{r}}$.

5. $64a^{12} - 27n^9$ by $4a^4 - 3n^3$; $16x^6y^{10n} - \frac{1}{64}m^8z^{20n}$ by $4x^3y^{5n} - \frac{1}{8}m^4z^{10n}$.

6. $a^8x^{8p} - b^8y^{8m}$ by $a^2x^{2p} - b^2y^{2m}$; $32x^{10} - 243y^{15}$ by $2x^2 - 3y^3$.

7. $x^{-\frac{4}{5}} - y^{-\frac{4}{5}}$ by $x^{-\frac{1}{5}} - y^{-\frac{1}{5}}$; $x^3 - y^3$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$; $ax^{\frac{5}{2}} - b^{\frac{5}{3}}y$ by $a^{\frac{1}{5}}x^{\frac{1}{2}} - b^{\frac{1}{3}}y^{\frac{1}{5}}$.

35. It may be shown by actual division that

$$\frac{a^2-b^2}{a+b} = a-b;\ \frac{a^4-b^4}{a+b} = a^3-a^2b+ab^2-b^3;\ \text{and so on.}$$

Hence, in general, it will be found that,

The difference of any two equal even powers of two numbers is divisible by the sum of the numbers.

In each of the above quotients we observe the laws are the same as in I. and III., Art. 34; also,

VI. *The signs are alternately + and −.*

Hence, the principle may be applied to different classes of examples as in Art. 34. Thus, in general,

If a and b are coefficients, $a\,x^p + b\,y^m$ divides $a^n\,x^{np} - b^n\,y^{nm}$ (n being *any even* and *positive* integer; also m and p may be integral, fractional, or negative), since the dividend is the difference between the nth powers of $a\,x^p$ and $b\,y^m$.

Note. The difference of the squares of two numbers is always divisible by the sum and also by the difference of the numbers. Thus, $a^6 - b^8$ is divisible by $a^3 \pm b^4$. In general, $a^{2n} - b^{2m}$ is divisible by $a^n \pm b^m$ when n and m are integral. This is the converse of Art. 26.

Exercise 28.

Divide by inspection:

1. $625\,a^4\,x^4 - 81\,m^4\,n^4$ by $5\,a\,x + 3\,m\,n$; $x^8 - b^{12}$ by $x^2 + b^3$; $x^8 - 1$ by $x + 1$.

2. $x^{\frac{2}{3}} - y^{\frac{4}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{2}{3}}$; $256\,x^4 - 10000$ by $4\,x + 10$; $x^{6m} - y^{4n}$ by $x^{3m} + y^{2n}$.

3. $x^{4m} - y^{4n}$ by $x^m + y^n$; $\frac{1}{16}\,x^2 - .0016\,y^6$ by $\frac{1}{2}\,x^{\frac{1}{2}} + .2\,y^{\frac{3}{2}}$; $a^{10} - b^{10}$ by $a + b$

4. $729\,a^{12} - 64\,b^{18}$ by $3\,a^2 + 2\,b^3$; $a^2\,x^{2n} - b^4\,y^{4m}$ by $a\,x^n + b^2\,y^{2m}$.

5. $a^{-\frac{4}{5}} - x^{-\frac{4}{5}}$ by $a^{-\frac{1}{5}} + x^{-\frac{1}{5}}$; $a^4\,x^{-\frac{4}{3}} - b^{\frac{1}{2}}\,y^{-1}$ by $a\,x^{-\frac{1}{3}} + b^{\frac{1}{8}}\,y^{-\frac{1}{4}}$.

6. $x^{\frac{3}{2}n} - y^{\frac{5}{2}m}$ by $x^{\frac{1}{4}n} + y^{\frac{5}{12}m}$; $81\,a^{-\frac{1}{3}n}\,x - \frac{1}{16}\,b^{\frac{2}{3}n}\,y^{-\frac{3}{2}m}$ by $3\,a^{-\frac{1}{12}n}\,x^{\frac{1}{4}} + \frac{1}{2}\,b^{\frac{1}{6}n}\,y^{-\frac{3}{8}m}$.

36. It may be shown by actual division that:

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2;\ \frac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4;\ \text{and so on.}$$

Hence, in general, it will be found that,

The sum of any two equal odd powers of two numbers is divisible by the sum of the numbers.

In each of the above quotients we observe that the laws are the same as in Art. 35.

Hence, the principle may be applied to all the different classes of examples as in Art. 34. Thus, in general,

If a and b are coefficients, $a\,x^p + b\,y^m$ divides $a^n x^{np} + b^n y^{nm}$ (n being *any odd* and *positive* integer, also m and p are integral, fractional, or negative), since the dividend is the sum of the nth powers of $a\,x^p$ and $b\,y^m$.

Exercise 29.

Divide by inspection:

1. $x^7 + y^7$ by $x + y$; $x^{-5} + y^{-5}$ by $x^{-1} + y^{-1}$; $1024\,x^5 + 243\,y^5$ by $4\,x + 3\,y$.

2. $128\,x^{21} + 2187\,y^{14}$ by $2\,x^3 + 3\,y^2$; $243\,x^{15} + 32\,y^{10}$ by $3\,x^3 + 2\,y^2$.

3. $x^{14n} + y^{21m}$ by $x^{2n} + y^{3m}$; $k^{21a} + m^{35n}$ by $k^{3a} + m^{5n}$; $a^{11} + b^{11}$ by $a + b$.

4. $m\,n + x\,y$ by $m^{\frac{1}{5}}n^{\frac{1}{5}} + x^{\frac{1}{5}}y^{\frac{1}{5}}$; $x^{\frac{5}{2}} + y^{\frac{15}{2}}$ by $x^{\frac{1}{2}} + y^{\frac{3}{2}}$; $x^{-\frac{7}{5}} + y^{-7}$ by $x^{-\frac{1}{5}} + y^{-1}$.

5. $a\,x^{\frac{7}{2}} + b^{\frac{5}{3}}y$ by $a^{\frac{1}{7}}x^{\frac{1}{2}} + b^{\frac{5}{21}}y^{\frac{1}{7}}$; $(\frac{2}{3})^{\frac{3}{5}} + (\frac{3}{2})$ by $(\frac{2}{3})^{\frac{1}{5}} + (\frac{3}{2})^{\frac{1}{3}}$; $a^5 + b^{10}$ by $a + b^2$.

Note. Since a^6 and b^6 are odd powers of a^2 and b^2, $a^6 + b^6$ is divisible by $a^2 + b^2$. a^{10} and b^{10} are the 5th powers of a^2 and b^2, $a^{10} + b^{10}$ is divisible by $a^2 + b^2$. Also, a^9 and b^9 are the *third* powers of a^3 and b^3. Therefore, $a^9 + b^9$ is divisible by $a^3 + b^3$

6. $a^{12} + b^{12}$ by $a^4 + b^4$; $x^6 + 1$ by $x^2 + 1$; $x^{12} + 1$ by $x^4 + 1$; $a^{27} + b^{27}$ by $a^9 + b^9$.

7. $x^{10} + y^{10}$ by $x^2 + y^2$; $x^{15} + y^{15}$ by $x^5 + y^5$; $64 + x^6$ by $2^2 + x^2$.

8. $64\,x^6 + 729\,y^6$ by $2^2\,x^2 + 9\,y^2$; $x^{10} + \frac{1}{1024}$ by $x^2 + (\frac{1}{2})^2$; $a^{24} + b^{24}$ by $a^8 + b^8$.

9. $a^{18} + b^{18}$ by $a^6 + b^6$ and by $a^2 + b^2$; $\frac{1}{729}\,x^6 + \frac{1}{64}\,y^6$ by $\frac{1}{9}\,x^2 + \frac{1}{4}\,y^2$.

10. $a^{36} + b^{36}$ by $a^{12} + b^{12}$ and by $a^4 + b^4$; $729\,x^6 + 1$ by $9\,x^2 + 1$.

11. $x^{42} + y^{42}$ by $x^6 + y^6$ and by $x^{14} + y^{14}$. **Query.** Is it divisible by $x^2 + y^2$? Why? $a^{15} + b^{18}$ by $a^5 + b^6$; $a^{27} + b^{21}$ by $a^9 + b^7$; $a^{15} + b^{15}$ by $a^3 + b^3$.

12. $x^{54} + y^{54}$ by $x^{18} + y^{18}$ and by $x^6 + y^6$. **Query.** Is it divisible by $x^2 + y^2$? Why? $a^9 + b^{12}$ by $a^3 + b^4$; $a^6\,b^9 + m^{27}\,n^{36}$ by $a^2\,b^3 + m^9\,n^{12}$; $a^{15}\,b^{25} + m^{30}\,n^{10}$ by $a^3\,b^5 + m^6\,n^2$.

Find an exact divisor and the quotient for each of the following, by inspection:

13. $8 + a^3$; $a^6 - b^6$; $8 - x^3$; $x^4 - 81$; $a^{15} - b^{12}$; $81\,a^{12} - 16\,b^8$; $a^4 - 625$; $a^9 - b^9$.

14. $x^{20} - y^{15}$; $m^5 + x^5$; $x^{12} - y^{12}$; $x^6 - 1$; $a^{-12} - b^{-12}$; $a^6\,x^{6p} - b^6\,y^{6p}$; $x^5 - 32$; $16\,a^4 - 81$.

15. $32\,a^5 - b^5$; $81\,a^8 - 16\,b^2$; $1 - y^7$; $a^3\,x^3 + 1000$; $a^4\,x^4 - 1$; $a^5 + m^5\,x^5$; $x^2\,y^2 - 81\,a^2$.

16. $32\,a^{10} - 243\,b^{15}$; $c^{10}\,x^{10n} - a^{10}\,x^{10m}$; $a^{9n} + b^{9m}$; $a^9\,x^{9n} + b^9\,y^{9m}$; $c^7\,x^{7p} + b^7\,y^{7n}$.

17. $x^{-6n} + y^{-6n}$; $8x^3y^3 + 729$; $a^{12}y^{12p} - b^{12}y^{12n}$; $c^8x^{8p} - b^8y^{8n}$, $x^4 - 1296$; $a^{15n} - b^{10n}$.

18. $128x^{21} + 2187y^{14}$; $256x^{12} - 81y^8$; $a^{5m}b^{5n} + x^{5r}y^{5s}$; $1 + 128x^{14}$; $a^{-8n} - b^{12n}$; $x^{-\frac{1}{2}}y^{-\frac{1}{2}} - 1$.

19. $x^{4n}y^{-\frac{1}{3}} - a^{\frac{1}{3}}y^{-1}$; $a^{-\frac{5}{2}}x^{-\frac{5}{2}} + 1$; $\frac{8}{27}a^nx^{\frac{1}{3}c} + \frac{64}{125}b^{-m}y^{-1}$; $\frac{1}{32}x^{-\frac{5}{3}n} - .00032\,y^{-5m}$.

20. $b^{\frac{5}{3}}c^n + .00243\,xy^{-1}$; $256\,ax^{-\frac{5}{2}n} - .0081\,b^{-\frac{2}{3}m}$; $2^{-\frac{4}{5}n}a^{\frac{1}{5}} - 3^{\frac{2}{3}m}b^{-\frac{1}{2}}$.

Queries. How divide a monomial by a monomial? Prove it. How prove the method for dividing a polynomial by a polynomial? In Art. 35, the sign of the last term of the quotient is $-$, while in Art. 36, the sign of the last term of the quotient is $+$. Why is this? What is the product of a^5 and a^{-3}? Prove it. What of a^m and a^{-n}? Prove it.

Miscellaneous Exercise 30.

Divide:

1. $a^3b^{-3} + \frac{5}{12}a^2b^{-2} + \frac{39}{16}$ by $\frac{1}{3}ab^{-2} + \frac{1}{2}b^{-1}$; $x^{-1} + y^{-1}$ by $x^{-\frac{1}{3}} + y^{-\frac{1}{3}}$.

2. $a^{-5} + 5a^{-4}b^{-1} + 10a^{-3}b^{-2} + 10a^{-2}b^{-3} + 5a^{-1}b^{-4} + b^{-5}$ by $a^{-1} + b^{-1}$.

3. $2a^2 - a^{\frac{1}{2}} - 2a + 1$ by $1 - a^{\frac{1}{2}}$; $x - y$ by $x^{\frac{1}{4}} - y^{\frac{1}{4}}$.

4. $(a - b - c)^n - (a - b - c)^{n-m} - (a - b - c)^m$ by $(a - b - c)^{n+m}$.

5. $2x^3 + 2y^3 + 2z^3 - 6xyz$ by $(x - y)^2 + (y - z)^2 + (z - x)^2$; $(x^3 - y^3)^2$ by $(x^2 + xy + y^2)^2$.

6. $(x^2 - 2yz)^3 - 8y^3z^3$ by $x^2 - 4yz$; $(x + 2y)^3 + (y - 3z)^3$ by $x + 3(y - z)$.

7. $x^{\frac{5}{2}} - x^{\frac{3}{2}} y + x y^{\frac{3}{2}} - 2 x^{\frac{1}{2}} y^2 + y^{\frac{5}{2}}$ by $x^{\frac{3}{2}} - x y^{\frac{1}{2}} + x^{\frac{1}{2}} y - y^{\frac{3}{2}}$.

8. $2 x^{3n} - 6 x^{2n} y^n + 6 x^n y^{2n} - 2 y^{3n}$ by $x^n - y^n$.

9. $x^{m+n} y^n - 4 x^{m+n-1} y^{2n} - 3 x^{m+n-2} y^{3n} + 6 x^{m+n-3} y^{4n}$ by $x^m - 3 x^{m-1} y^n - 6 x^{m-2} y^{2n}$.

10. $a^{3n} - 3 a^{2n} b^n + 3 a^n b^{2n} - b^{3n}$ by $a^n - b^n$; $8 a^{-x} - 8 a^x + 5 a^{3x} - 3 a^{-3x}$ by $5 a^x - 3 a^{-x}$.

11. $6 a^{8a+2} - 23 a^{7a+1} + 18 a^{6a} - a^{5a-1} - 3 a^{4a-2} + 4 a^{3a-3} - a^{2a-4}$ by $2 a^{4a+1} - 5 a^{3a} - 2 a^{2a-1} + a^{a-2}$.

12. $4 a^{\frac{2}{3}} - 8 a^{\frac{1}{3}} - 5 + 10 a^{-\frac{1}{3}} + 3 a^{-\frac{2}{3}}$ by $2 a^{\frac{5}{12}} - a^{\frac{1}{12}} - 3 a^{-\frac{1}{4}}$.

13. $6 x^{x+3} - 5 x^{x+2} - 6 x^{x+1} + 19 x^x - 21 x^{x-1} + 4 x^{x-2}$ by $2 x^3 + x^2 - 4 x$.

14. $6 m^{x-n+2} + m^{x-n+1} - 22 m^{x-n} + 19 m^{x-n-1} - 4 m^{x-n-2}$ by $3 m^{3-n} - 4 m^{2-n} + m^{1-n}$.

15. $6 x^{x+n+2} + x^{x+n+1} - 9 x^{x+n} + 11 x^{x+n-1} - 6 x^{x+n-2} + x^{x+n-3}$ by $2 x^{n+2} + 3 x^{n+1} - x^n$.

16. $a^{mn} - a^n b^{(n-1)m} - a^{(m-1)n} b^m + b^{mn}$ by $a^n - b^m$.

Find an exact divisor and the quotient of the following, by inspection:

17. $8 x^3 + 1$; $16 - 81 a^4$; $64 a^3 - 8 b^3$; $a^3 + 1000$; $a^6 - 64$; $m^5 - n^5$; $1 - 8 y^3$; $a^7 b^7 - 1$.

18. $x^4 - x^{-4}$; $\frac{32}{243} a^n x^{5p} + \frac{1024}{3125} b^{5k}$; $x^{12n} + y^{-12m}$; $x^{35} - y^{25}$; $x^{-12m} - y^{12n}$; $x^{9m} + y^{9n}$.

19. $8 x^6 - 27 y^{-9}$; $64 a^{12} - 27 n^{-9}$; $243 a^{\frac{n}{2}} + 32$; $c^5 x^{5n} - a^5 y^{5m}$; $\frac{16}{81} a^{4n} - .0016 b^{\frac{1}{2}n}$; $2^{9n} a^{12n} + 3^{6n}$.

CHAPTER VII.

EVOLUTION.

37. Evolution is the operation of finding one of the equal factors of a number or expression. Evolution is the inverse of involution.

By Art. 27, $(2a)^2 = 4a^2$; $(2a)^3 = 8a^3$; $(2a)^4 = 16a^4$; etc.

$2a$ is called the second or square root of $4a^2$ because it is one of the two equal factors of $4a^2$; it is the third or cube root of $8a^3$ because it is one of the three equal factors of $8a^3$; etc. Hence, in general,

A **Root** is one of the equal factors of the number or expression.

Roots are indicated by means of fractional exponents, the denominators of which show the root to be taken.

Thus, $(a)^{\frac{1}{2}}$ means the second or square root of a; $(a)^{\frac{1}{3}}$ means the third or cube root of a; $(a^5)^{\frac{1}{6}}$ means the sixth root of a^5. In general, $(a^m)^{\frac{1}{n}}$ means the nth root of a^m.

Roots are also indicated by means of the *root* sign, or *radical* sign, $\sqrt{\ }$.

Thus, $\sqrt{a}$ means the square root of a; $\sqrt[3]{a}$ means the cube root of a; $\sqrt[n]{a^m}$ means the nth root of a^m.

The **Index** is the number written in the opening of the radical sign to show what root is sought, and corresponds to the denominator of the fractional exponent. When no index is written, the square root is understood.

Note. $\sqrt[n]{a}$ or $a^{\frac{1}{n}}$ is defined, when n is a positive integer, as one of the n equal factors of a; so that if $\sqrt[n]{a}$ be taken n times as a factor, the resulting product is a; that is, $(\sqrt[n]{a})^n$ or $\left(a^{\frac{1}{n}}\right)^n = a$.

Similarly, $\left(\sqrt[mn]{a}\right)^{mn}$ or $\left(a^{\frac{1}{mn}}\right)^{mn} = a$.

38. The sign, $\pm$ or $\mp$, is sometimes used and is called the *double sign;* it indicates that we may take either the sign $+$ or the sign $-$. Thus, $a \pm b$ is read *a plus or minus b.*

By Art. 27, $(+a)^4 = a^4$; $(-a)^4 = a^4$; $(+a)^5 = a^5$; $(-a)^5 = -a^5$.

Therefore, $(a^4)^{\frac{1}{4}} = \pm a$; $(+a^5)^{\frac{1}{5}} = a$; $(-a^5)^{\frac{1}{5}} = -a$. Hence, in general,

Even roots of any number are either positive or negative.

Odd roots of a number have the same sign as the number itself.

Since no even power of a number can be negative, it follows that,

An even root of a negative number is impossible.

Such roots can only be indicated, and are called *imaginary.* Thus, $(-a^2)^{\frac{1}{2}}$, $\sqrt[4]{-6}$, $\sqrt[8]{-1}$, and $\sqrt[10]{-a^6}$, are imaginary.

EXAMPLE 1. Find the square root of $9a^6b^4c^2$.

Solution. Since, to square a monomial, we multiply the exponent of each factor by 2, to extract the square root we must divide the exponent of each factor by 2. The two equal factors of 9 are 3×3, or 3^2. Dividing the exponent of each factor by 2, we have $3a^3b^2c$. Since the even root of a positive number is either positive or negative, the sign of the root is either plus or minus.

$$\therefore \sqrt{9a^6b^4c^2} = \pm 3a^3b^2c.$$

EXAMPLE 2. Find the fifth root of $-32a^{10}x^{5n}$.

Solution. Since, to raise a monomial to the fifth power, we multiply the exponent of each factor by 5, to extract the fifth root we must divide the exponent of each factor by 5. The equal factors of 32 are $2 \times 2 \times 2 \times 2 \times 2$, or 2^5. Dividing the exponent of each

factor by 5, we have $2a^2x^n$. Since the odd roots of a number have the same sign as the number itself, the sign of the root is minus.

$$\therefore \sqrt[5]{-32a^{10}x^{5n}} = -2a^2x^n. \text{ Hence, in general,}$$

To find any Root of a Monomial. *Resolve the numerical coefficient into its prime factors, each factor being written with its highest exponent, divide the exponent of each factor by the index of the required root, and take the product of the resulting factors. Give to every even root of a positive expression the sign $\pm$, and to every odd root of any expression the sign of the expression itself.*

Note. Any root of a fraction is found by taking the required root of each of its terms. Thus, $\sqrt[3]{\frac{8}{27}} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}} = \frac{2}{3}$. In general, $\sqrt[r]{\frac{m}{n}} = \frac{\sqrt[r]{m}}{\sqrt[r]{n}}$.

Exercise 31.

Find the value of the following expressions:

1. $\sqrt{25x^2y^4}$; $\sqrt[3]{-8a^6b^3x^9}$; $\sqrt[3]{-125a^3b^6}$; $\sqrt[4]{81a^{16}b^{20}}$.
2. $(-343a^4b^{-6})^{\frac{1}{3}}$; $(\frac{64}{81}x^4y^6z^8)^{\frac{1}{2}}$; $(-x^{10}y^{15})^{\frac{1}{5}}$; $\sqrt[4]{81x^4y^{-4}}$.
3. $\sqrt[9]{x^{18}}$; $(121x^{12}y^2)^{\frac{1}{2}}$; $\sqrt{25a^2b^2}$; $(16a^{-8}b^8)^{\frac{1}{4}}$.
4. $(-243a^{5n}b^{10n})^{\frac{1}{5}}$; $(-64m^3n^6x^9)^{\frac{1}{3}}$; $(m^{50}n^{30})^{\frac{1}{10}}$.
5. $(-32a^{10}y^{-5})^{\frac{1}{5}}$; $\sqrt[5]{32a^{15}x^{-10}}$; $(625a^8b^{16}c^4)^{\frac{1}{4}}$.
6. $(512a^6b^3c^{15}d^{-3})^{\frac{1}{3}}$; $\sqrt{64a^{-6}b^4}$; $\sqrt[7]{128}$; $\sqrt[5]{-32a^{15}}$.
7. $\sqrt[5]{a^{10m}}$; $\sqrt[4]{\frac{x^4}{y^4}}$; $\sqrt[3]{27n^6m^9}$; $\sqrt[7]{a^{14}b^{21}c^{-7}}$; $(a^{2m}b^{3m})^{\frac{1}{m}}$.
8. $\sqrt[n]{x^{mn}y^{an}}$; $(2^na^{2n}b^{4n}x^{3n})^{\frac{1}{n}}$; $\sqrt{81x^{2n}y^{2m+4}}$; $\sqrt[3]{-8x^{3n-6}y^{6n+9}}$.

9. $\sqrt[5]{243\,x^{10}y^{-10}z}$; $\sqrt[3]{\frac{343}{1000}\,a^3b^{15}c^{18}x^{-21}}$; $(-125\,a^6b^{12}p^{-15})^{\frac{1}{3}}$.

10. $\sqrt[4]{16\,x^{4n}\,y^{12n}\,z^8}$; $\left(-\frac{64}{125}\,m^{-\frac{9}{2}}\,n^{-\frac{3}{2}}\right)^{\frac{1}{3}}$; $\sqrt[m]{a^m\,b^{3m}\,c^{-m}}$.

Simplify:

11. $\sqrt{\frac{25}{16}\,a^2\,b^{\frac{1}{3}}\,c^{-\frac{1}{2}}} + \left(\frac{8}{27}\,a^3\,b^{\frac{1}{2}}\,c^{-\frac{3}{4}}\right)^{\frac{1}{3}} - \left(\frac{81}{16}\,a^4\,b^{\frac{2}{3}}\,c^{-1}\right)^{\frac{1}{4}}$
$- \sqrt[5]{\frac{32}{5^5}\,a^5\,b^{\frac{5}{6}}\,c^{-\frac{5}{4}}}$.

Express the nth roots of:

12. $3 \times 7 \times 4$; $5\,x^n\,y^{2n}$; $3\,a^{\frac{n}{2}}b^3$; $\left(a^{-\frac{n}{4}}\right)^3$; $(x\,y)^n$; $x^{m-1}y^{2n}$; $x^n\,y^{-n}$; $(a\,b^3\,c)^{6n}$; $(x+y)^n\,(x-y)^{-n}$.

Express by means of exponents:

13. $\sqrt[n]{a\,b^3\,c^n}$; $\sqrt[m]{a\,(x-y^n)^m}$; $\sqrt[2p]{a^p\,x^n}$; $\sqrt[2n]{(x+y)^m}$.

Queries. If n and p in the last two parts of Ex. 13 are integral, what signs should the roots have? Why? When should the first two roots have the double sign?

39. By Art. 28, $\quad (a+b)^2 = a^2 + 2\,a\,b + b^2$.

Therefore, $(a^2 + 2\,a\,b + b^2)^{\frac{1}{2}} = a + b$.

By observing the manner in which $a + b$ may be obtained from $a^2 + 2\,a\,b + b^2$, we shall be led to a general method for finding the square root of any polynomial.

Process.		$a^2 + 2\,a\,b + b^2\,(a + b$
First term of the root squared,		a^2
First remainder,		$2\,a\,b + b^2$
Trial divisor,	$2\,a$	
Complete divisor,	$2\,a + b$	
Complete divisor $\times\, b$,		$2\,a\,b + b^2$

Explanation. The square root of the first term is a, which is the first term of the required root. Subtracting its square from the given expression, the remainder is $2\,a\,b + b^2$, or b times $2\,a + b$.

Since the first term of the remainder is twice the product of the first and last terms of the root, and we have found the first term; therefore, divide $2ab$ by *twice* the first term of the root already found, or $2a$. The result will be the *second* term b of the required root. Adding b to the trial divisor gives the complete divisor, $2a+b$. Multiplying by b and subtracting, there is no remainder.

By Art. 28, $(a+b+c)^2 = a^2+2ab+b^2+2ac+2bc+c^2$.

Therefore, $(a^2+2ab+b^2+2ac+2bc+c^2)^{\frac{1}{2}} = a+b+c$.

Process. $a^2+2ab+b^2+2ac+2bc+c^2\,(a+b+c$

First term of root squared, $\underline{a^2}$

First remainder, $2ab+b^2+2ac+2bc+c^2$

First trial divisor, $2a$

First complete divisor, $\underline{2a+b}$

First complete divisor $\times\, b$, $\underline{2ab+b^2}$

Second remainder, $2ac+2bc+c^2$

Second trial divisor, $2a+2b$

Second complete divisor, $\underline{2a+2b+c}$

Second complete divisor $\times\, c$, $\underline{2ac+2bc+c^2}$

Explanation. Proceeding as before, the first two terms of the root are found to be $a+b$. To find the last term of the root, take *twice* the terms of the root already found for the second trial divisor. Dividing $2ac$ by the first term, the result c will be the third term of the required root. Adding this to the trial divisor, gives the entire divisor. Multiplying by c and subtracting there is no remainder. We have actually squared the root and subtracted the square from the given expression. Hence, in general,

To find the Square Root of any Polynomial. Arrange the terms according to the powers of one letter. Find the square root of the first term. This will be the first term of the required root. Subtract its square from the given expression. Divide the first term of the remainder by twice the root already found. The quotient will be the next term of the root. Add the quotient to the divisor. Multiply the complete divisor by this term of the root, and subtract the product from the remainder. For the next trial divisor, take two times the terms of the root already found. Continue in this manner until there is no remainder.

EXAMPLE. Find the square root of $4a - 10a^3 + a^6 + 4a^5 + 1$.

Arranging according to the ascending powers of a, we have,

Process.		$1+4a-10a^3+4a^5+a^6 \,(1+2a-2a^2-a^3$
First term of root squared,		1
First remainder,		$4a-10a^3+4a^5+a^6$
First trial divisor,	2	
First complete divisor,	$2+2a$	
$2a$ times first complete divisor,		$4a^2+4a$
Second remainder,		$-4a^2-10a^3+4a^5+a^6$
Second trial divisor,	$2+4a$	
Second complete divisor,	$2+4a-2a^2$	
$-2a^2$ times second complete divisor,		$-4a^2-8a^3+4a^4$
Third remainder,		$-2a^3-4a^4+4a^5+a^6$
Third trial divisor,	$2+4a-4a^2$	
Third complete divisor,	$2+4a-4a^2-a^3$	
$-a^3$ times third complete divisor,		$-2a^3-4a^4+4a^5+a^6$

Note. The student should notice that the sum of the several subtrahends is the square of the root, and that he has actually squared the root and subtracted the square from the given expression.

Exercise 32.

Find the square roots of:

1. $y^4 - 4y^3 + 6y^2 - 4y + 1$; $9a^4 - 12a^3 - 2a^2 + 4a + 1$.

2. $4a^6 - 12a^5b - 11a^4b^2 + 58a^3b^3 - 17a^2b^4 - 70ab^5 + 49b^6$.

3. $x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$.

4. $8a + 4 + a^4 + 4a^3 + 8a^2$.

5. $9 + x^6 + 30x - 4x^5 + 13x^2 + 14x^4 - 14x^3$.

6. $6ab^2c - 4a^2bc + a^2b^2 + 4a^2c^2 + 9b^2c^2 - 12abc^2$.

7. $49a^4 - 28a^3 - 17a^2 + 6a + \frac{9}{4}$.

8. $4x^2 + 9y^2 + 25a^2 + 12xy - 30ay - 20ax$.

9. $m^6 - 6am^5 + 15a^2m^4 - 20a^3m^3 + 15a^4m^2 - 6a^5m + a^6$.

10. $1 - 2a + 3a^2 - 4a^3 + 5a^4 - 4a^5 + 3a^6 - 2a^7 + a^8$.

11. $9m^2 - 6mn + 30mx + 6my + n^2 - 10nx - 2ny + 25x^2 + 10xy + y^2$.

12. $x^6 + 15x^2y^4 + 15x^4y^2 + y^6 - 6xy^5 - 20x^3y^3 - 6x^5y$.

13. $49x^2y^2 - 24xy^3 - 30x^3y + 25x^4 + 16y^4$.

14. $x^6 - 6x^5 + 17x^4 - 34x^3 + 46x^2 - 40x + 35$.

15. $4 - 16a^{\frac{2}{3}} + 16a^{\frac{4}{3}} + 12b^{\frac{1}{2}} - 24a^{\frac{2}{3}}b^{\frac{1}{2}} + 9b$.

16. $\frac{1}{9}x^4 - \frac{2}{3}x^3y + \frac{4}{3}x^2y^2 - xy^3 + \frac{1}{4}y^4$; $x^{4n} - 6x^{3n} + 5x^{2n} + 12x^n + 4$.

17. $25x^{\frac{4}{3}} + 16 - 30x - 24x^{\frac{1}{3}} + 49x^{\frac{2}{3}}$.

18. $9x^{-2} + 12x^{-1}y^2 - 6x + 4y^4 - 4x^2y^2 + x^4$.

40. Since the square root of an expression is either + or –, the square root of $a^2 + 2ab + b^2$ is either $a + b$ or $-a - b$. In the process of finding the square root of $a^2 + 2ab + b^2$, we begin by taking the square root of a^2, and this is either $+a$ or $-a$. If we take $-a$, and continue the work as in Art. 39, we get for the root $-a - b$. Also, the square root of $a^2 - 2ab + b^2$ is either $a - b$ or $-a + b$. This is true for every *even* root. Hence, *the signs of all the terms of an even root may be changed, and the number will still be the root of the same expression.* Thus, last process Art. 39, if -1 be taken for the square root of 1 we shall arrive at the result $-1 - 2a + 2a^2 + a^3$.

41. Square Root of Numerical Numbers. The method for extracting the square root of arithmetical numbers is based upon the algebraic method.

Since the square root of 100 is 10, of 10000 is 100, etc., it follows that the integral part of the square root of numbers less than 100 has *one* figure, of numbers between 100 and 10000 *two* figures, and so on. Hence,

If a point be placed over every second figure in any number, beginning with units' place, the number of points will show the number of figures in the square root.

Thus, the square root of $3\dot{2}4\dot{9}4\dot{7}$ has three figures; the square root of $\dot{4}4\dot{1}$ has two figures. If the given number contains decimals, the number of decimal places in the square root will be one half as many as in the given number itself. Thus, if 2.39 be the square root, the number will be $\dot{5}.7\dot{1}2\dot{1}$; if .239 be the root, the number will be $\dot{0}.05\dot{7}1\dot{2}1$; if 10.321 be the root, the number will be $\dot{1}0\dot{6}.5\dot{2}3\dot{0}4\dot{1}$. Hence,

The number of points to the left of the decimal point will show the number of integral places in the root, and the number of points to the right will show the number of decimal places.

EXAMPLE 1. Find the square root of 45796.

		$a+b+c=214$
Process.	$\dot{4}5\dot{7}9\dot{6}$	$(200+10+4=214$
The square of a or 200,	40000	
First remainder,	5796	
First trial divisor, $2a$, or 400		
First complete divisor, $2a+b$, or 410		
First complete divisor $\times\, b$, or 10,	4100	
Second remainder,	1696	
Second trial divisor, $2a+2b$, or 420		
Second complete divisor, $2a+2b+c$, or 424		
Second complete divisor $\times\, c$, or 4,	1696	

Explanation. There will be three figures in the root. Let $a+b+c$ denote the root, a being the value of the number in the hundreds' place, b of that in the tens' place, and c the number in the units' place.

Then a must be the greatest multiple of 100 whose square is less than 45796, this is 200. Subtract a^2, or the square of 200 from the given number. Dividing the first remainder by $2a$, or 400, gives 10

for the value of b. Add this to 400, multiply the result by 10 and subtract. Dividing the second remainder by $2a + 2b$, or 420, gives 4 for the value of c. Adding this to 420, multiplying and subtracting, there is no remainder. Hence, 214 is the required root; because we have actually squared it and subtracted this square from the given number and found no remainder. The student should observe that the sum of the several subtrahends is the square of the root.

Example 2. Find the square root of 17.3 to four decimal places.

Process.		$\dot{1}7.\dot{3}0\dot{0}0\dot{0}0\dot{0}0$ (4.1593....
Square of 4,		16
First remainder,		130
First trial divisor,	8	
First complete divisor,	81	
First complete divisor multiplied by 1,		81
Second remainder,		4900
Second trial divisor,	82	
Second complete divisor,	825	
Second complete divisor multiplied by 5,		4125
Third remainder,		77500
Third trial divisor,	830	
Third complete divisor,	8309	
Third complete divisor multiplied by 9,		74781
Fourth remainder,		271900
Fourth trial divisor,	8318	
Fourth complete divisor,	83183	
Fourth complete divisor multiplied by 3,		249549
Fifth remainder,		22351

. . . .

Let the student formulate a method for arithmetical square root from what has been demonstrated.

Notes: 1. If the trial divisor is not contained in the remainder, annex 0 to the root, also to the divisor, then annex the next period and divide.

2. Should it be found that after completing the trial divisor, it gives a product greater than the remainder, the quotient is too large, and a less quotient must be taken.

3. If the last remainder is not a perfect square, annex periods of ciphers and proceed as before.

4. The square root of a fraction may be found by taking the square root of its terms, or by first reducing it to a decimal.

Exercise 33.

Find the square roots of:

1. 33124; 41.2164; $\frac{169}{225}$; $\frac{625}{1369}$; .099225; 1.170724.

2. .30858025; 5687573056; 943042681.

Find the square root to four decimal places of:

3. .081; .9; .001; .144; $\frac{13}{52}$; .00028561; 3.25; 20.911.

42. By Art. 29, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Therefore, $(a^3 + 3a^2b + 3ab^2 + b^3)^{\frac{1}{3}} = a + b$.

By observing the manner in which $a + b$ may be obtained from $a^3 + 3a^2b + 3ab^2 + b^3$, we shall be led to a general method for finding the cube root of any compound expression.

Process.		$a^3+3a^2b+3ab^2+b^3\,(a+b$
First term of the root cubed,		a^3
First remainder,		$3a^2b+3ab^2+b^3$
Trial divisor, or 3 times the square of a,	$3a^2$	
3 times the product of a and b,	$3ab$	
Second term of the root squared,	b^2	
Complete divisor,	$3a^2+3ab+b^2$	
Complete divisor $\times\ b$,		$3a^2b+3ab^2+b^3$

Explanation. The cube root of the first term is a, which is the first term of the required root. Subtracting its cube from the given expression, the remainder is $3a^2b+3ab^2+b^3$, or b *times* $3a^2+3ab+b^2$. Since the first term of the remainder is three times the product of the square of the first term of the root multiplied by the last term, divide $3a^2b$ by *three* times the square of the first term of the root already found. The result will be the *second* term b of the required root. Adding to the trial divisor three times the product of the first and second terms of the root, and the square of the second term, gives the complete divisor, or $3a^2 + 3ab + b^2$. Multiplying by b and subtracting, there is no remainder.

Since the cube of $a+b+c$ is $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$, the cube root of $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$ is $a+b+c$.

Process.

		$a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3\,(a+b+c$
First term of the root cubed,		a^3
First remainder,		$3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$
First trial divisor, or 3 times the square of a,	$3a^2$	
3 times the product of a and b,	$3ab$	
Second term of the root squared,	b^2	
First complete divisor,	$3a^2+3ab+b^2$	
First complete divisor $\times$ b,		$3a^2b+3ab^2+b^3$
Second remainder,		$3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$
Second trial divisor, 3 times the square of root already found,	$3(a+b)^2=3a^2+6ab+3b^2$	
3 times the product of $a+b$ and c,	$3ac+3bc$	
Third term c of the root squared,	c^2	
Second complete divisor,	$3a^2+6ab+3b^2+3ac+3bc+c^2$	
Second complete divisor $\times$ c,		$3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$

Explanation. Proceeding as before, the first two terms of the root are found to be $a + b$. To find the last term of the root, take *three* times the square of the terms of the root already found for the second trial divisor, and divide $3a^2c$ by the first term. The result will be the third term of the required root. Adding to the second trial divisor three times the product of $a + b$ and c, and the square of c, gives the second complete divisor. Multiplying by c and subtracting, there is no remainder. Observe that the sum of the several subtrahends is the cube of the root, and that we have actually cubed the root and subtracted the cube from the given expression. Hence, in general,

To find the Cube Root of any Polynomial. Arrange the terms according to the powers of one letter. Find the cube root of the first term. This will be the first term of the required root. Subtract its cube from the given expression. Divide the first term of the remainder by three times the square of the root already found. The quotient will be the next term of the root. Add to the trial divisor three times the product of the first and second terms of the root, and the square of the second term. Multiply the complete divisor by this term of the root, and subtract the product from the remainder. For the next trial divisor, take three times the square of the root already found. Continue in this manner until there is no remainder or an approximate root found.

A **Term** may be a figure, or a letter, or a combination of figures and letters, or of letters only, produced by multiplication or division, or both.

Thus, in the algebraic expression $5+2a^3b^4-a+\frac{ab^n}{x^2y^m}$; 5, $2a^3b^4$, a, $\frac{ab^n}{x^2y^m}$ are terms.

An **Algebraic Expression** is a representation of a number by any combination of algebraic symbols.

EXAMPLE. Find the cube root of $27a-8a^{\frac{3}{2}}-36+36a^{\frac{5}{2}}-12a^{-1}-54a^{\frac{7}{2}}+9a^{-\frac{5}{2}}+27a^{\frac{9}{2}}+a^{-6}-6a^{-\frac{7}{2}}$.

The work is conveniently arranged as follows :

Process.

$$\left(3a^{\frac{3}{2}}-2a^{\frac{1}{2}}+a^{-2}\right.$$

$$27a^{\frac{9}{2}}-54a^{\frac{7}{2}}+36a^{\frac{5}{2}}-8a^{\frac{3}{2}}+27a-36-12a^{-1}+9a^{-\frac{5}{2}}-6a^{-\frac{7}{2}}+a^{-6}$$

First term of the root cubed, $27a^{\frac{9}{2}}$

First remainder, $-54a^{\frac{7}{2}}+36a^{\frac{5}{2}}-8a^{\frac{3}{2}}+27a-36-12a^{-1}+9a^{-\frac{5}{2}}-6a^{-\frac{7}{2}}+a^{-6}$

First trial divisor, 3 times the square of $3a^{\frac{3}{2}}$, $27a^3$

3 times the product of $3a^{\frac{3}{2}}$ and $-2a^{\frac{1}{2}}$, $-18a^2$

Second term $-2a^{\frac{1}{2}}$ of the root squared, $4a$

First complete divisor, $27a^3-18a^2+4a$

First complete divisor $\times -2a^{\frac{1}{2}}$, $-54a^{\frac{7}{2}}+36a^{\frac{5}{2}}-8a^{\frac{3}{2}}$

Second remainder, $27a-36-12a^{-1}+9a^{-\frac{5}{2}}-6a^{-\frac{7}{2}}+a^{-6}$

Second trial divisor, $3(3a^{\frac{3}{2}}-2a^{\frac{1}{2}})^2=27a^3-36a^2+12a$

3 times the product of $3a^{\frac{3}{2}}-2a^{\frac{1}{2}}$ and a^{-2}, $9a^{-\frac{1}{2}}-6a^{-\frac{3}{2}}$

Third term of the root, or a^{-2} squared, a^{-4}

Second complete divisor, $27a^3-36a^2+12a+9a^{-\frac{1}{2}}-6a^{-\frac{3}{2}}+a^{-4}$

Second complete divisor $\times a^{-2}$, $27a-36-12a^{-1}+9a^{-\frac{5}{2}}-6a^{-\frac{7}{2}}+a^{-6}$

Exercise 34.

Find the cube roots of:

1. $x^6 - 3x^5 + 5x^3 - 3x - 1$; $x^3 - 3ax^5 + 5a^3x^3 - 3a^5x - a^6$.

2. $8x^6 + 48ax^5 + 60a^2x^4 - 80a^3x^3 - 90a^4x^2 + 108a^5x - 27a^6$.

3. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

4. $27a^6 - 54a^5b + 9a^4b^2 + 28a^3b^3 - 3a^2b^4 - 6ab^5 - b^6$.

5. $8x^6 + 12x^5 - 30x^4 - 35x^3 + 45x^2 + 27x - 27$.

6. $216 + 342x^2 + 171x^4 + 27x^6 - 27x^5 - 109x^3 - 108x$.

7. $a^3 - 3a^2b - b^3 + 8c^3 + 6a^2c - 12abc + 6b^2c + 12ac^2 - 12bc^2 + 3ab^2$.

8. $1 - 3x + 6x^2 - 10x^3 + 12x^4 - 12x^5 + 10x^6 - 6x^7 + 3x^8 - x^9$.

9. $8x^6 - 36x^5y + 114x^4y^2 - 207x^3y^3 + 285x^2y^4 - 225xy^5 + 125y^5$.

10. $a^3 + 6a^2b - 3a^2c + 12ab^2 - 12abc + 3ac^2 + 8b^3 - 12b^2c + 6bc^2 - c^3$.

11. $x^6 + 3x^5y - 3x^4y^2 - 11x^3y^3 + 6x^2y^4 + 12xy^5 - 8y^6$.

12. $204x^4y^2 - 144x^5y + 8y^6 - 36xy^5 - 171x^3y^3 + 64x^6 + 102x^2y^4$.

43. **Cube Root of Numerical Numbers.** The method for extracting the cube root of arithmetical numbers is based upon the algebraic method.

Since the cube root of 1000 is 10 ; of 1000000 is 100, etc., it follows that the integral part of the cube root of numbers less than 1000 has *one* figure, of numbers between 1000 and 1000000 *two* figures, and so on. Hence,

If a point be placed over every third figure in any number, beginning with units' place, the number of points will show the number of figures in the cube root.

Thus, the cube root of $27\dot{4}62\dot{5}$ has two figures; the cube root of $10\dot{9}21\dot{5}35\dot{2}$ has three figures.

If the given number contains decimals, the number of decimal places in the cube root will be one third as many as in the given number itself. Thus, if 1.11 be the cube root, the number will be $\dot{1}.36\dot{7}63\dot{1}$; if .111 be the root, the number will be $\dot{0}.00\dot{1}36\dot{7}63\dot{1}$; if 11.111 be the root, the number will be $\dot{1}37\dot{1}.70\dot{0}96\dot{0}63\dot{1}$. Hence,

The number of points to the left of the decimal point will show the number of integral places in the root, and the number of points to the right will show the number of decimal places.

EXAMPLE 1. Find the cube root of 778688.

			$a+b=92$
Process.		$7\dot{7}868\dot{8}$	($90+2=92$
The cube of a, or 90,		729000	
First remainder,		49688	
First trial divisor $3\,a^2$, or $3\,(90)^2$	$=24300$		
3 times the product of a and b, or $3\times90\times2=$	540		
Second term b of the root squared, 2^2	$=\quad 4$		
First complete divisor,	24844		
First complete divisor $\times$ b, or 2,		49688	

Explanation. There will be two figures in the root. Let $a+b$ denote the root, a being the value of the number in tens' place, and b the number in units' place. Then a must be the greatest multiple of 10 whose cube is less than 778688, this is 90. Subtract a^3, or the cube of 90, from the given number. Dividing the remainder by $3\,a^2$, or 24300, gives 2 for the value of b. Add to the trial divisor $3\,a\,b$, or 540, and b^2, or 4, for the complete divisor. Multiplying by 2 and subtracting, there is no remainder. Hence, 92 is the required

root, because we have actually cubed it and subtracted this cube from the given number and found no remainder.

EXAMPLE 2. Find the cube root of 897.236011125.

Process.		897.236011125 (9.645
Cube of 9,		729
First remainder,		168236
First trial divisor, 3 times $(90)^2$	= 24300	
3 times the product of 90 and 6,	1620	
6 squared,	36	
First complete divisor,	25956	
First complete divisor multiplied by 6,		155736
Second remainder,		12500011
Second trial divisor, 3 times $(960)^2$	= 2764800	
3 times the product of 960 and 4,	11520	
4 squared,	16	
Second complete divisor,	2776336	
Second complete divisor multiplied by 4,		11105344
Third remainder,		1394667125
Third trial divisor, 3 times $(9640)^2$	= 278788800	
3 times the product of 9640 and 5,	144600	
5 squared,	25	
Third complete divisor,	278933425	
Third complete divisor multiplied by 5,		1394667125

Let the student formulate a method for arithmetical cube root from what has been demonstrated.

Note. The notes in Art. 41 are equally applicable to cube root, except that in Note 1 *two* ciphers must be annexed to the divisor instead of one.

Exercise 35.

Find the cube roots of:

1. 74088; 34012.224; .244140625.

2. $\frac{2515456}{32768}$; .000152273304.

Find to three places of decimals the cube roots of:

3. .64; .08; 8.21; .3; .008; $\frac{2}{3}$; $\frac{7}{27}$.

44. Since $\quad a^{\frac{1}{4}} = a^{\frac{1}{2\times 2}} = \left(a^{\frac{1}{2}}\right)^{\frac{1}{2}} = \sqrt{a^{\frac{1}{2}}} = \sqrt{\sqrt{a}},$

The fourth root is the square root of the square root.

Since $\quad a^{\frac{1}{6}} = a^{\frac{1}{2\times 3}} = \left(a^{\frac{1}{2}}\right)^{\frac{1}{3}} = \sqrt[3]{a^{\frac{1}{2}}} = \sqrt[3]{\sqrt{a}},$

The sixth root is the cube root of the square root. Hence,

When the root indices are composed of factors, the operation is performed by successive extraction of simpler roots.

Note. It is suggested that the teacher use the remainder of this article at his discretion.

We may find the *fifth, seventh, eleventh,* or any root of an expression or arithmetical number if desired, by using the form for completing the divisor. Thus,

To find the fifth root.

Form, $(a+b)^5 = a^5 + (5\,a^4 + 10\,a^3b + 10\,a^2b^2 + 5\,ab^3 + b^4)\,b$
Trial divisor, $5\,a^4$.
Complete divisor, $\quad (5\,a^4 + 10\,a^3b + 10\,a^2b^2 + 5\,ab^3 + b^4)$.

To find the seventh root.

Form, $(a+b)^7 = a^7 + (7a^6+21a^5b+35a^4b^2+35a^3b^3+21a^2b^4+7ab^5+b^6)\,b$.
Trial divisor, $7\,a^6$.
Complete divisor, $(7a^6+21a^5b+35a^4b^2+35a^3b^3+21a^2b^4+7ab^5+b^6)$.

EXAMPLE. Find the fifth root of 36936242722357.

Process.

		$3693\dot{6}242\dot{7}2235\dot{7}$ (517
$a^5 = 5^5 =$		3125
First remainder,		56862427
First trial divisor $= 5\,a^4$ (a considered as 5 tens) $= 5\,(50)^4 =$	31250000	
$10\,a^3b$ (b considered as 1 unit) $= 10(50)^3 \times 1 =$	1250000	
$10\,a^2b^2 = 10 \times (50)^2 \times (1)^2 =$	25000	
$5\,a\,b^3 = 5 \times (50) \times (1)^3 =$	250	
$b^4 = (1)^4 =$	1	
First complete divisor,	32525251	
First complete divisor multiplied by 1,		32525251
Second remainder,		2433717622357
Second trial divisor $= 5\,a^4$ (a considered as 51 tens) $= 5 \times (510)^4 =$	338260050000	
$10\,a^3b$ (b considered as 7 units) $= 10 \times (510)^3 \times (7) =$	9285570000	
$10a^2b^2 = 10 \times (510)^2 \times (7)^2 =$	127449000	
$5\,a\,b^3 = 5 \times (510) \times (7)^3 =$	874650	
$b^4 = (7)^4 =$	2401	
Second complete divisor,	347673946051	
Second complete divisor multiplied by 7,		2433717622357

Miscellaneous Exercise 36.

Express the nth roots of:

1. $a\,b^{\frac{1}{2}}c^{-\frac{n}{2}}$; $5^{2n}x^{3n}(x-y+z^n)^{4n} \times 2^n(x+y^n)^{4n} \times 4^n(x-y^n)^n$.

2. Simplify $4\,a\,(8\,a\,x\,y)^{\frac{1}{2}} - 5\,x^{\frac{1}{2}}\,y^{\frac{1}{2}}\sqrt[3]{2^{2\frac{1}{2}}\,a^4 \times a^{\frac{1}{2}}}$.

Find the square roots of:

3. $9\,x + 10 + x^{-1} - 4\,x^{-\frac{1}{2}} + 12\,x^{\frac{1}{2}}$.

4. $28 - 24\,a^{-\frac{n}{2}} - 16\,a^{\frac{n}{2}} + 9\,a^{-n} + 4\,a^n$.

5. $16x^{6n} + 16x^{7n} - 4x^{8n} - 4x^{9n} + x^{10n}$.

6. $x^3y^{-\frac{2}{3}} - 4x^{\frac{3}{2}}y^{-\frac{1}{3}} + 6 - 4x^{-\frac{3}{2}}y^{\frac{1}{3}} + x^{-3}y^{\frac{2}{3}}$.

7. $6acx^5 + 4b^2x^4 + a^2x^{10} + 9c^2 - 12bcx^2 - 4abx^7$.

8. $\frac{1}{4}x^4 + 4x^2 + \frac{1}{3}ax^2 + \frac{1}{9}a^2 - 2x^3 - \frac{4}{3}ax$.

9. $a^{2n} \mp 2a^nx^m + x^{2m}$; $a \pm 2a^{\frac{1}{2}}x^{\frac{1}{2}} + x$.

Find the cube roots of:

10. $60x^2y^4 + 48xy^5 - 27x^6 + 108x^5y - 90x^4y^2 + 8y^6 - 80x^3y^3$.

11. $24x^{4m}y^{2n} + 96x^{2m}y^{4n} - 6x^{5m}y^n + x^{6m} - 96x^my^{5n} + 64y^{6n} - 56x^{3m}y^{3n}$.

12. $15x^{-4} - 6x^{-1} - 6x^{-5} + 15x^{-2} + 1 + x^{-6} - 20x^{-3}$.

13. $8x^3 - 4x^2y^2 + \frac{2}{3}xy^4 - \frac{1}{27}y^6$.

14. $\frac{3}{2}a^{-\frac{1}{2}} - 6a^{-1} - \frac{1}{8} + 8x^{-\frac{3}{2}} - \frac{27}{2}a^{-3} + 27a^{-\frac{1}{2}} + 54a^{-\frac{7}{2}} + \frac{9}{4}a^{-\frac{3}{2}} + 36a^{-\frac{5}{2}} - 18a^{-2}$.

Find the sixth roots of:

15. $1215a^4 - 1458a^5 + 135a^2 - 540a^3 - 18a + 1 + 729a^6$.

16. $x^6 + y^6 - 6x^2y^5 + 15x^2y^4 - 6x^5y + 15x^4y^2 - 20x^3y^3$.

17. $160a^3 + 240a^4 + 60a^2 + 192a^5 + 64a^6 + 12a + 1$.

18. 2985984; 262144.

Find the eighth roots of:

19. $a^8 + 28a^2 + 8a + 1 + 56a^3 + 70a^4 + 8a^7 + 56a^5 + 28a^6$.

20. $(a^4 + b^4 - 2ab^3 + 3a^2b^2 + 2a^3b)^4$.

21. Find the 5th root of 36936242722357.

22. Find the 7th root of 12311715481324093344.

Extract the following roots:

23. $\sqrt{(a^4 + 19\,a^2 + 25 - 6\,a^3 - 30\,a)}$.

24. $\sqrt[4]{[x^4 - 2\,(m+n)\,x^3 + (m^2 + 4\,m\,n + n^2)\,x^2 - 2\,m\,n\,(m+n)\,x + m^2\,n^2]^2}$.

25. $[25\,a^2 - 20\,a\,b + 4\,b^2 + 9\,c^2 - 12\,b\,c + 30\,a\,c]^{\frac{1}{2}}$.

26. $[27\,a^6 - 54\,a^5 + 63\,a^4 - 44\,a^3 + 21\,a^2 - 6\,a + 1]^{\frac{1}{3}}$.

27. $\sqrt{(x^{2m} + 2\,x^{m+n} - 2\,x^{m+1} + x^{2n} - 2\,x^{n+1} + x^2)}$.

28. $[a^6 - 12\,a^5 + 60\,a^4 - 160\,a^3 + 240\,a^2 - 192\,a + 64]^{\frac{1}{6}}$.

29. $[(a+b)^{6m}x^3 + 6a^nc(a+b)^{4m}x^2 + 12a^{2n}c^2(a+b)^{2m}x + 8a^{3n}c^3]^{\frac{1}{3}}$.

30. $\sqrt{(x^{2n} + 2\,x^{2n-1} + 3\,x^{2n-2} + 2\,x^{2n-3} + x^{2n-4})}$.

31. $\sqrt[3]{(8 - 12\,a^{3n-1} + 6\,a^{6n-2} - x^{9n-3})}$.

Queries. What signs are given to *even* and *odd* roots? Why? What principles govern the signs of roots? Upon what principle is the method for finding the root of a monomial based? How derive the method for finding the square root of any polynomial? Why divide the first term of the remainder by *twice* the terms of the root already found for the next term of the root? Why add the quotient to the trial divisor for the complete divisor? How derive the method for finding the cube root of any polynomial? Why divide the first term of the remainder by *three* times the square of the root already found for the next term of the root? Why add to the trial divisor *three* times the product of the terms of the root already found by the next term, and the square of the next term, for the complete divisor?

CHAPTER VIII.

USE OF ALGEBRAIC SYMBOLS.

45. SYMBOLS of operation are used to indicate that algebraic operations are to be performed.

Thus, $m+(a-b)$ indicates that $a-b$ is to be added to m; $m-(a-b)$ indicates that $a-b$ is to be subtracted from m. Performing the operations, we have,

$$m+(a-b)=m+a-b;$$
$$m-(a-b)=m-a+b. \quad \text{Hence,}$$

A plus sign before a symbol of aggregation shows that the enclosed terms are to be added to what precedes; as this operation does not change the signs, the removal of the symbol does not affect the signs. Removing one preceded by a minus sign changes the sign of each enclosed term.

$$\begin{aligned}
\text{Thus, } & a-2b-[4a-6b-\{3a-c+(5a-2b-\overline{3a-c+2b})\}] \\
&=a-2b-[4a-6b-\{3a-c+(5a-2b-3a+c-2b)\}] \\
&=a-2b-[4a-6b-\{3a-c+(2a-4b+c)\}] \\
&=a-2b-[4a-6b-\{3a-c+2a-4b+c\}] \\
&=a-2b-[4a-6b-\{5a-4b\}] \\
&=a-2b-[4a-6b-5a+4b] \\
&=a-2b-[-2b-a] \\
&=a-2b+2b+a \\
&=2a
\end{aligned}$$

Explanation. Remove the vinculum, subtract and unite like terms; then remove the parenthesis and unite like terms; now remove the brace, subtract and unite like terms; finally, removing the bracket, subtracting and uniting like terms, we have $2a$.

Exercise 37.

Simplify:

1. $2a - [3b + (2b - c) - 4c + \{2a - (3b - \overline{c - 2b})\}]$.

2. $a - b + c - (a + b - c) - (c - b - a)$.

3. $x^4 - [4x^3 - \{6x^2 - (4x - 1)\}] - (x^4 + 4x^3 + 6x^2 + 4x + 1)$.

4. $-10(x + y) - [z + x + y - 3\{x + 2y - (z + x - y)\}] + 4z$.

5. $a - [5b - \{a - (5c - \overline{2c - b - 4b}) + 2a - (a - \overline{2b + c})\}]$.

6. $-5\{a - 6[a - (b - c)]\} + 60\{b - (c + a)\}$.

7. $2a - (3b + 2c) - [5b - (6c - 6b) + 5c - \{2a - (c + 2b)\}]$.

8. $3x - \{y - [y - (x + y) - \{-y - (y - \overline{x - y})\}]\}$.

9. $3a - [2b + \overline{a - b}] + [3b - \overline{2a + b}]$.

10. $\{(x - 2y + xy) - (x - y + z)\} - \{x - (x - y + xy)\}$.

11. $\frac{2}{3}a - [\frac{3}{2}a - \{\frac{1}{3}a - (2a - \overline{5a + 6})\} - (\frac{5}{6}a - 3)]$.

12. $\frac{3}{8}\{\frac{4}{3}(a - b) - 8(b - c)\} - \{\frac{1}{2}(b - c) - \frac{1}{3}(c - a)\} - \frac{1}{2}\{c - a - \frac{2}{3}(a - b)\}$.

13. $5\{a - 2[a - 2(a + x)]\} - 4\{a - 2[a - 2(a + x)]\}$.

14. $a + 2b - \{6a - [3b + (8x - \overline{2 + by} - x + 4a)] - 3b\} + 2(1 + \frac{1}{2}a - 4b)$.

15. $2(\frac{3}{2}b - \frac{5}{2}a) - 7[a - 6\{2 - 5(a - b)\}]$.

16. $-\frac{3}{5}\{-\frac{2}{3}[-4(-x^{\frac{1}{2}})]\} + \frac{5}{3}\{-\frac{3}{5}(-x^{\frac{1}{2}})\}$.

17. $-\frac{2}{3}\{-[-(a - b)]\} + \{-\frac{2}{5}[-(a - b)]\}$.

18. $5\{a-2[b-3(c+d)]\}-4\{a-3[b-4(c-d)]\}$.

19. $(a-1)(a-2)-3a(a+3)+2\{(a+2)(a+1)-3\}$.

20. $\{xz-(x-y)(y+z)\}-y[y-(x-z)]$.

21. $(a+b+c+d)^2+(a-b-c+d)^2+(a-b+c-d)^2+(a+b-c-d)^2$.

22. $x^2-\{2xy-[-(x-\{y-z\})(x-\{y-z\})+2xy]-4yz\}-(y+z)^2$.

23. $a(a+1)(a+2)(a+3)-6(2a-\frac{1}{6})-(a^2-3a+1)^2$.

24. $5n\{(x-y)a-bz\}-2n\{x(a-b)-ay\}-\{3ax-(5z-2x)b\}n$.

25. $(x^2+y^2)n-(x+y)(x\{n-y\}-y\{n-x\})$.

26. $2a\sqrt{x}-3m-[by^{\frac{1}{2}}-6n+(x^{\frac{1}{2}}-2\sqrt{x})a]+b\times\sqrt{y}$.

27. $(9m^2n^2-4n^4)(m^2-n^2)-\{3mn-2n^2\}\{3m(m^2+n^2)-2n(n^2+3mn-m^2)\}n$.

28. $m^2(m^2+n^2)^2-2m^2n^2(m+n)(m-n)-(m^3-n^3)^2$.

29. $\frac{4}{3}(\frac{1}{2}x+\frac{1}{3}y)(\frac{1}{2}x-\frac{1}{3}y)-(\frac{1}{3}x-\frac{2}{3}y)^2-\frac{2}{9}(x^2-\frac{8}{3}y^2)$.

30. $\frac{16}{9}(\frac{1}{2}x^2+\frac{1}{3}y^2)(\frac{1}{2}x^2-\frac{1}{3}y^2)-(\frac{2}{3}x-3)(\frac{2}{3}x+3)(\frac{4}{9}x^2-9)+(\frac{2}{3}y-3)(\frac{2}{3}y+3)(\frac{4}{9}y^2-9)$.

The use of symbols of aggregation aid in shortening the work in certain cases in division. Thus,

$$\begin{array}{l|l|l}
a+(b+c) & (b+c)a^2+(b^2+bc+c^2)a-(b+c)bc & (b+c)a-bc \\
 & \underline{(b+c)a^2+(b^2+2bc+c^2)a} & \\
 & +(-bc)a-(b+c)bc & \\
 & \underline{+(-bc)a-(b+c)bc} &
\end{array}$$

Divide:

31. $(b+c)a^2+(b^2+3bc+c^2)a+(b+c)bc$ by $a+b+c$.

32. $(a+b)^2-6(a+b)-27$ by $(a+b)+3$.

33. $(x+y)^3+3(x+y)^2z+3(x+y)z^2+z^3$ by $(x+y)^2+2(x+y)z+z^2$.

34. $(x+y)^2-2(x+y)z+z^2$ by $x+y-z$.

35. $(a+b)^3+1$ by $a+b+1$.

46. The converse operation of enclosing any number of terms of an expression in a symbol of aggregation is important.

$$a+m-c+b-n=a+m-c+(b-n).$$
$$a-m-c+b-n=a-(m+c)+(b-n).$$
$$ax^2-ny+bx^3-cy^4=(ax^2-ny)+(bx^3-cy^4).$$
$$xy-ax-by+ab=(xy-by)-(ax-ab).$$

Hence, when the signs + and − indicate operation:

(1) *Any number of terms may be enclosed in a symbol of aggregation preceded by the sign +, without changing the sign of each term.*

(2) *Any number of terms may be enclosed in a symbol of aggregation preceded by the sign −, if the sign of each term be changed.*

The terms may be enclosed in various ways. Thus,

$$am+an-ax-bx+cy-dz=(am-ax)+[an-bx]+\{cy-dz\},$$
or, $am+an-ax-bx+cy-dz=(am+an-ax)-\{bx-cy+dz\}$,
or, $am+an-ax-bx+cy-dz=(am+an)-(ax+bx)+(cy-dz)$. Etc.

If a factor is common to each term within a symbol of aggregation, it may be placed outside as a multiplier. Thus,
$$ax^3+bx^2-bx^3+dx^2=(ax^3-bx^3)+(bx^2+dx^2)=x^3(a-b)+x^2(b+d).$$

Note. An expression consisting of three or more terms may be raised to a given power by inspection, by first changing it to the form of a binomial. Thus, $(a+b+c-d)^4=[(a+b)+(c-d)]^4=$ etc.

Exercise 38.

Bracket the last three terms so that each bracket shall be preceded by a $-$ sign:

1. $x^4 - a x^3 - 5 x^2 + 2$; $m^5 + 3 m^3 + 3 - 6 m^2$.

2. $3 x - 2 y + 5 z - 4 n$; $a^3 b^3 - 2 a^2 b^5 + b^4 - a b^6$.

3. $4x + 3 a x^2 - 6 x^3 - 5 c y + y$; $x^2 - y^2 - z^2 + ab + 3 ac$.

4. Express each of the above as binomials, and enclose the last two terms in an inner brace preceded by a $-$ sign.

Bracket the following in binomials, also in trinomials, each preceded by a $-$ sign:

5. $2 a b - 3 a y + 4 b z - 5 b x - 2 c d - 3$.

6. $a - 2 b + c z - d - 1 + z - x - 2 y + 2 m - n + p - 4 a b c$.

7. $2 x - 3 x y + 4 x^2 y^2 - 5 x^3 y^2 + x^2 y^3 - x y z$.

8. $x^5 + 3 a^4 - 4 a^3 - 3 a^2 + a - 1$; $-2 m - 3 n + 4 p - 5 x - 1 - 6 y$.

9. $a n + a b - a c - c x - a x - a y - 3 a b c + 3 x y z$.

10. Express the above six examples in trinomials, and enclose the last two terms in an inner bracket preceded by a $-$ sign.

11. Expand $(m + 2 n - x)^3$.

12. Simplify and bracket like powers of x in $2 b x^3 - a x - \{a x^2 - [b x - n x - \{a x^3 + 3 c x^2\}] - (a x^2 - 2 c x)\}$.

Queries. Why may a symbol of aggregation preceded by a + sign be removed without changing the signs of the enclosed terms? If a symbol of aggregation preceded by a $-$ sign be removed, why change the signs of the enclosed terms?

CHAPTER IX.

SIMPLE EQUATIONS.

47. $3x + 5 = 5x - 7$ is called an **Equation**. The *first member* or *first side* is $3x + 5$, and the *second member* or *second side* is $5x - 7$.

$x = x$, $14 = 14$, are called **Identities** or **Identical Equations**.

To **solve** an equation is to find the value of the unknown number.

The process of solving an equation depends upon the following axioms:

1. *If to equal numbers we add equal numbers, the sums are equal.*

2. *If from equal numbers we subtract equal numbers, the remainders are equal.*

3. *If equal numbers are multiplied by equal numbers, the products are equal.*

4. *If equal numbers are divided by equal numbers, the quotients are equal.*

EXAMPLE 1. Find the value of x in the equation $6x-11 = 3x+10$.

Solution. Subtracting $3x$ from each member of the equation (Axiom 2), we have $6x - 3x - 11 = 3x - 3x + 10$. Uniting like terms, $3x - 11 = 10$. Adding 11 to each member (Axiom 1), and uniting like terms, we get $3x = 21$. Dividing both members by 3 (Axiom 4) gives $x = 7$.

Proof. To verify this result, substitute 7 for x in the given equation. Then, $6 \times 7 - 11 = 3 \times 7 + 10$, or $31 = 31$, which is an identity. Hence, the value of x is 7.

EXAMPLE 2. Solve the equation $2(x-8)-3(9-x)+5(x-11) = 7-3(x-17)$.

Solution. Performing the indicated operations, and uniting like terms, $10x - 98 = 58 - 3x$. Adding $3x$ and 98 to each member of the equation, we have $10x + 3x - 98 + 98 = 58 + 98 - 3x + 3x$, or uniting like terms, $13x = 156$. Dividing both members by 13, $x = 12$.

Proof. Substitute 12 for x in the given equation.

Then, $2(12-8) - 3(9-12) + 5(12-11) = 7 - 3(12-17)$,

or, $8 + 9 + 5 = 7 + 15$,

or, $22 = 22$, an identity.

Therefore the value of x is 12.

EXAMPLE 3. Solve the equation $14 - x - 5(x-3)(x+2) + (5-x)(4-5x) = 45x - 76$.

Process. Simplify,	$64 - 25x = 45x - 76.$
Subtract $45x$,	$64 - 70x = -76.$
Subtract 64,	$-70x = -140.$
Divide by -70,	$x = 2.$

Notes: 1. To **verify**, that is, to *prove the truth of the result*, substitute the supposed value of the unknown number in the given equation and thus find if it satisfies its conditions.

2. In simplifying an equation the student should be careful to notice that when the sign $-$ precedes a term, in removing the symbol of aggregation, the sign of each term must be changed.

Exercise 39.

Solve the following equations:

1. $6x + 1 = 5x + 10$; $11 - 7x = 18x - 14$.
2. $32x - 22 = 14 + 65x$; $4x - 3x + 2 = 4x + 1$.
3. $2x + 3 = 16 - (2x - 3)$; $3(x-2) + 4 = 4(3-x)$.
4. $7(x - 18) = 3(x - 14)$; $7x + 6 - 3x = 56 + 2x$.
5. $15(x - 1) + 4(x + 3) = 2(x + 7)$.
6. $5 - 3(4 - x) + 4(3 - 2x) = 0$.

48. If we add the same number to each member of an equation, or subtract it from each member, the results are equal, each to each. Thus,

Consider the equation $x - b = a$. Adding b to each side, we get,

$$x = a + b.$$

Consider the equation $x + b = a$. Subtracting b from each side, we have $x = a - b$.

In each case b is transposed from one side to the other, but its sign is *changed*. Hence,

Any term may be transposed from one side of an equation to the other, provided its sign be changed.

EXAMPLE. Solve $(x+1)(x+2)(x+6) - (x-2)(x+2) = x^3 + 9x^2 + 4(7x-1) + (2-x)(3+x)$.

Process. Simplify, $x^3 + 8x^2 + 20x + 16 = x^3 + 8x^2 + 27x + 2$.
Transpose, $x^3 - x^3 + 8x^2 - 8x^2 + 20x - 27x = 2 - 16$.
Unite like terms, $-7x = -14$.
Divide by -7, $x = 2$.
Hence, in general,

To Solve a Simple Equation of one Unknown Number. *If necessary, simplify the equation. Transpose all the terms containing the unknown number to one side, and all other terms to the other side. Unite like terms, and divide both sides by the coefficient of the unknown number.*

Exercise 40.

Solve the following equations:

1. $12x - 20x + 13 = 9x - 259$; $336 + (3x - 11) = 2(5x - 5) + 8(97 - 7x)$.

2. $6x + 4x = 3x + 84$; $6x + 2(13 - x) = 3(17 - x)$.

3. $2(x-2)+18x=3(5+x)+6$; $30x+20x-15x+12x=2820$.

4. $9(x-1)+2(x-2)=10(2-x)$; $2(x+2)(x-4)=x(2x+1)-21$.

5. $6y-2(9-4y)+3(5y-7)=10y-(4+16y+35)$, and verify.

6. $2y-(4y-1)=5y-(y+1)$; $56+21x-8(2x-1)=62$.

7. $10[224-(x+192)]=7(28+3x)$; $9(7+9y)-4[9-(2-y)]=252y$.

8. $25x-19-[3-\{4x-3\}]=x-(x-5)$, and verify.

9. $20(2-x)+3(x-7)-2[x+9-3\{9-4(2-x)\}]=1$.

10. $(y-2)(7-y)+(y-5)(y+3)-2(y-1)+12=0$.

11. $4(y+5)^2-(2y+1)^2=3(y-5)+180$; $2.25x-1.25=3x+3.75$.

12. $.15y+1.575-.875y=.0625y$.

Query. In transposing, why change the signs?

49. Known Numbers are represented by the first letters of the alphabet, and by figures; as, a, b, $2c$, 6.

Unknown Numbers are usually represented by the last letters of the alphabet; as, x, y, z.

An **Equation** is a statement that two expressions represent the same number.

An **Identical Equation**, or an **Identity**, is one which is true for *all values* of the letters which enter into it; as, $(a+x)(a-x)=a^2-x^2$.

The **Roots** of an equation are the values of the unknown numbers.

The **Degree** of an equation is the power of the unknown number, and is determined by the greatest number of unknown factors in any term.

Thus, $x - y = 6$ is an equation of the *first* degree; $4x^2 + 5y = 3$ and $5xy + 2 = 3x$ are equations of the *second* degree.

A **Simple Equation** is an equation of the first degree.

Miscellaneous Exercise 41.

Solve the following equations:

1. $5(7+3y)-(2y-3)(1-2y)-(2y-3)^2+(5+y) = 0.$

2. $(2y+1)^3+(2y-1)^3 = 16y(y^2-4)+27$, and verify.

3. $1.5(26y-51)-12(1-3y)=78y-2[5y-2.5(1-.3y)].$

4. $.6x-.7x+.75x-.875x+15 = 0$; $.6y-(.18y-.05) = .2y + 4.45.$

5. $30z - 3[30z - (2z - 5)] = 5(2z - 57) - 50.$

6. $10(z + 10) - 18(3z - 4) + 5(3z - 2)(2z - 3) = 30z^2 - 16$, and verify.

7. $4.8y - 2(.72y - .05) = 1.6y + 8.9$; $.5x - .\dot{3}x - .25 = .25x - 1.$

8. $.2y - .1\dot{6}y = .6 - .\dot{3}$; $.5y - .\dot{2}y = .\dot{3}y - 15.$

9. $5.\dot{6}y + .25y + .\dot{3}y = y - 3$; $.\dot{6}y + .25 - .\dot{1}y = 1.\dot{8} - .75y - .\dot{3}.$

10. $3\{x - .25(x - 2) - .\dot{3}(3x + 12)\} = 41.$

CHAPTER X.

PROBLEMS LEADING TO SIMPLE EQUATIONS.

50. The beginner will find the model solutions of great benefit in forming statements, and he should give them careful consideration before attempting to solve any of the problems in each set.

Exercise 42.

1. A father is 35 and his son 8 years old. In how many years will the father be just twice as old as the son?

Solution. Let $x =$ the *number* of years required.

Then $x + 35 =$ the *number* of years in the father's age x years from now,

and $x + 8 =$ the *number* of years in the son's age x years from now.

By the conditions of the problem, at the expiration of x years twice the son's age, or $2(x + 8)$, equals the father's age, or $x + 35$. Hence, the equation $2(x + 8) = x + 35$, or $2x + 16 = x + 35$. Transposing and uniting like terms, $x = 19$.

2. One number exceeds another by 5, and their sum is 29. Find the numbers.

3. The difference of two numbers is 14, and their sum is 48. Find the numbers.

4. A father gave \$200 to his five sons, which they are to divide according to their ages, so that each elder son shall receive \$10 more than his next younger brother. Find the share of each.

5. A father is four times as old as his son; in 24 years he will only be twice as old. Find their ages.

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6. Divide 50 into two parts, so that three times the greater may exceed 100 by as much as 8 times the less falls short of 120.

Solution. Let $x =$ the greater part.

Then $50 - x =$ the less part,

and $3x =$ three times the greater part;

also, $8(50 - x) =$ eight times the less part.

But, $3x - 100 =$ the excess of three times the greater part over 100;

also, $120 - 8(50 - x) =$ the number that eight times the less lacks of 120.

By the conditions, $3x - 100 = 120 - 8(50 - x)$.

Therefore, $x = 36$, for the greater part,

and $50 - x = 14$, for the less part.

7. Twenty-three times a certain number is as much above 14 as 16 is above seven times the number. Find the number.

8. A is five years older than B. In 15 years the sum of their ages will be three times the present age of A. Find the age of each.

9. A is 25 years older than B, and A's age is as much above 20 as B's is below 85. Find their ages.

10. The sum of the ages of A and B is 30 years, and five years hence A will be three times as old as B. Find their ages.

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11. The difference between the squares of two consecutive numbers is 121. Find the numbers.

Solution. Let $x =$ the less number.
Then will $x + 1 =$ the greater number,
$x^2 =$ the square of the less number,
and $(x + 1)^2 =$ the square of the greater number.
Then $(x + 1)^2 - x^2 =$ the difference of the square numbers.
But $121 =$ the difference of the squares.
Hence, $(x + 1)^2 - x^2 = 121$.
Therefore, $x = 61$, the less number,
$x + 1 = 62$, the greater number.

12. Find three consecutive numbers whose sum is 27.

13. The difference of two numbers is 3, and the difference of their squares is 21. Find the numbers.

14. Find a number such that if 5, 15, and 35 be added to it, the product of the first and third results may be equal to the square of the second.

15. I sold a cow for $35 and half as much as I gave for it, and gained $10. Find the cost of the cow.

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16. A had four times as much money as B; but, after giving B $16, he had only two times as much as B. How much had each at first?

Solution. Let $x =$ the *number* of dollars that B had at first.
Then $4x =$ the *number* of dollars that A had at first.
But $4x - 16 =$ the *number* of dollars that A had after giving B $16,
and $x + 16 =$ the *number* of dollars B had after receiving $16 from A.
By the conditions, $4x - 16 = 2(x + 16)$.
Therefore, $x = 24$, the *number* of dollars that B had,
and $4x = 96$, the *number* of dollars that A had.

17. A father is 3 times as old as his son; four years ago the father was 4 times as old as his son then was. Find their ages.

18. One number is two times another; but if 50 be subtracted from each, one will be three times the other. Find the numbers.

19. A has \$26.20 and B has \$35.80. B gave A a certain sum; then A had four times as much as B. How much did A receive from B?

20. If 288 be added to a certain number, the result will be equal to three times the excess of the number over 12. Find the number.

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21. A farmer has grain worth \$0.60 per bushel, and other grain worth \$1.10 per bushel. How many bushels of each kind must be taken to make a mixture of 40 bushels worth \$0.90 a bushel?

Solution.

Let $x =$ the *number* of bushels required of the \$0.60 grain.

Then $40 - x =$ the *number* of bushels required of the \$1.10 grain;

and $\frac{60}{100}x =$ the *number* of dollars in the cost of the \$0.60 grain;

also, $1.10(40-x) =$ the *number* of dollars in the cost of the \$1.10 grain.

Hence, $\frac{60}{100}x + 1.10(40 - x) =$ the *number* of dollars in the total cost of the mixture.

But the cost of the mixture is to be \$36. Hence,

$$\frac{60}{100}x + 1.10(40 - x) = 36.$$

Therefore, $x = 16$, the *number* of bushels of the \$0.60 kind,

and $40 - x = 24$, the *number* of bushels of the \$1.10 kind.

22. A merchant has two kinds of vinegar: one worth \$0.35 a quart and the other \$1.25 a gallon. From these he made a mixture of 63 gallons, worth \$1.30 a gallon. How many gallons did he take of each kind?

23. A merchant has a mixture of 88 pounds of 13 and 11 cent sugar, which he sells at $13\frac{3}{4}$ cents per pound. How many pounds of each kind are there?

24. I bought 24 pounds of tea of two different kinds, and paid for the whole \$9. The better kind cost \$0.65 per pound, and the poorer kind \$0.35 per pound. How many pounds were there of each kind?

25. A grocer having 75 pounds of tea worth \$0.90 a pound, mixed with it so much tea at \$0.50 a pound that the combined mixture was worth \$0.80 a pound. How much did he add?

Remarks. No general method can be given for the solution of problems.

The beginner will find that his principal difficulty in solving a problem consists in forming the equation of conditions, and in order to overcome this, much will depend upon his skill and ingenuity.

The *statement* of a problem consists in translating its conditions into algebraic symbols and ordinary language. Many times the beginner fails to form a correct statement, because he does not understand what is meant by the ordinary language of the problem. If he cannot assign a consistent meaning to the words, it will be impossible for him to express their meaning in algebraic symbols. It often happens that the words appear to be susceptible of more than one meaning. In such cases the student should express the meaning that seems most reasonable in algebraic symbols, and obtain the result to which it will lead. Should such result be inadmissible, the student should try another meaning of the words.

The student *must depend upon his own powers*, and should he at times be perplexed, he must not be discouraged, since nothing but patience and practice can overcome the difficulties and give him readiness and certainty in solving problems. He must study the meaning of the language of the problem, to ascertain the *unknown numbers* in it. There may be several such numbers, but oftentimes a little skilful manipulation will enable one to express all of the unknown numbers in terms of some one of them. Select the one by which this can be most easily done and represent it by some one of the final letters of the alphabet.

Among the following problems no doubt the beginner will find

some which he can readily solve by arithmetic, or by guessing and trial; he may thus be led to undervalue the power of algebra, and to regard its aid as unnecessary. In reply, as the student advances he will find that by the aid of algebra he can solve not only *all* of these problems, without any uncertainty or *guessing*, but those which would be exceedingly difficult, if not altogether impossible, if he depended upon arithmetical processes alone.

26. A's age is six times B's, and fifteen years hence A will be three times as old as B. Find their ages.

27. A is three times as old as B, and 12 years since he was five times as old. Find B's age.

28. A father has three sons; his age is 60, and the joint ages of the sons is 46. How long will it be before the joint ages of the sons will be equal to that of the father?

29. If you walk 10 miles, then travel a certain distance by train, and then twice as far by coach, and the whole journey is 70 miles, how far will you travel by coach?

30. A is twice as old as B, and seven years ago their united ages amounted to as many years as now represent the age of A. Find their ages.

31. After 136 quarts had been drawn out of one of two equal casks, and 80 gallons out of the other, there remained just three times as much in one cask as in the other. Find the contents of each cask.

32. Find the number whose double increased by 1.2 exceeds 3.65 by as much as the number itself is less than 8.65.

33. Find three consecutive numbers such that if they be diminished by 10, 17, and 26, respectively, their sum will be 10.

34. Two consecutive numbers are such that one fourth of the less exceeds one fifth of the greater by 1. Find the numbers.

35. There are two consecutive numbers such that one fifth of the greater exceeds one tenth of the less by 3. Find them.

36. Find a number such that the sum of its half and its fourth shall exceed the sum of its fifth and its tenth by 45.

37. Find a number such that the sum of its half and its fifth shall exceed the difference of its fourth and its tenth by 110.

38. If a watch and chain are worth $185, and the watch lacks $19 of being worth two times the cost of the chain, find the cost of each.

39. If silk costs 6 times as much as linen, and I buy 22 yards of silk and 28 yards of linen at a cost of $52, find the cost of each per yard.

40. A man gave 17 boys $3.31, giving to some 13 cents each and to the rest 23 cents each. How many received 23 cents?

41. I paid a bill of $1.03 with 39 pieces of money, some 3-cent and the rest 5-cent pieces. How many of each did it take?

42. A son earns 37 cents per day less than his father, and in 8 days the father earns $6.08 more than the son earns in 5 days. Find the daily wages of each.

43. How many 10-cent pieces and how many 25-cent pieces must be taken so that 95 pieces shall make $12.35?

44. Divide \$112 into two parts, so that the number of five-cent pieces in one may equal the number of three-cent pieces in the other.

45. A sum of money consists of dollars, twenty-five-cent pieces, and dimes, and amounts to \$29.50. The number of coins is 55. There are twice as many dimes as quarters. How many are there of each kind?

46. A sum of £8 17*s.* is made up of 124 coins, consisting of florins and shillings. How many are there of each?

47. A bill of £4 5*s.* was paid in crowns, half-crowns, and shillings. The number of half-crowns used was four times the number of crowns and twice the number of shillings. How many were there of each?

48. A bill of £$48\frac{1}{2}$ was paid with guineas and half-crowns, and 12 more half-crowns than guineas were used. How many were there of each?

49. A company of 84 persons consists of men, women, and children. There are three times as many men as women, and five times as many women as children. How many are there of each?

50. The sum of three numbers is 263. The first is 3 times the second, and the third is 23 more than 5 times the sum of the other two. Find the numbers.

51. A farmer wishes to mix 660 bushels of feed, containing oats, corn, rye, and barley, so that the mixture may contain two times as much corn as oats, three times as much rye as corn, and four times as much barley as rye. How many bushels of each should be used?

52. Divide $2590 into two such parts that the first at 7% simple interest for 8 years may amount to the same sum as the second in 5 years at 8%.

Note. The character % is sometimes used for the term "*per cent.*" *Per cent* is used by ellipsis for *rate per cent*. Thus, an allowance of 7 on a hundred is at a *rate* of .07, and the *rate per cent* is 7.

53. $330 is invested in two parts, on one of which 15% is gained, and on the other 8% is lost. The total amount returned from the investment is $345. Find the investment.

54. A man has $7585. He built a house, and put the rest out at simple interest for 18 months; 40% of it at 5% and the remainder at 6%. The income from both investments is $211.26. Find the cost of the house.

55. In a certain weight of gunpowder the saltpetre was 4 pounds less than half the weight, the sulphur 5 pounds more than a fifth, and the charcoal 3 pounds more than a tenth. Find the number of pounds of each.

56. A company of 266 persons consists of men, women, and children. The men are 14 more in number than the women; the children 34 more than the men and women together. How many are there of each?

57. I bought 16 yards of cloth, and if I had bought one yard less for the same money, each yard would have cost $0.25 more. Find the cost per yard of the cloth.

58. A and B, 85 miles apart, set out at the same time to meet each other; A travels 5 miles an hour and B 4 miles an hour. How far will each have travelled when they meet?

59. \$330 is loaned for nine months in two parts; on one 15 % per annum is gained, and on the other 8 % per annum is lost. The total amount from the loan is \$364.25. Find the amount in each loan.

60. A boy has a certain sum of money, he borrowed as much more, and spent 12 cents; he again borrowed as much as he had left, and spent 12 cents; again he borrowed as much as he had left, and spent 12 cents; after which he had nothing left. How much money had he at first?

61. A carriage, horse, and harness are worth \$720. The carriage is worth eight tenths of the value of the horse, and the harness six tenths of the difference between the value of the horse and carriage. Find the value of each.

62. A boy sold half an apple more than half his apples. Again he sold half an apple more than half his remaining apples. A third time he repeated the process; and he had sold all his apples. How many apples had he?

Algebra is the science which treats of algebraic numbers and the symbols of relation.

Algebra, like arithmetic, is a science which treats of numbers. In arithmetic the numbers are positive and represented by figures. In algebra the letters of the alphabet or figures are used to represent numbers, and they may be positive or negative, real or imaginary.

Algebra enables us to prove general theorems respecting numbers, and also to express those theorems briefly.

CHAPTER XI.

FACTORING.

51. A **Factor** is one of the makers of a number.

Thus, since 5 with the aid of 4 and by the process of multiplication makes 20, 5 is a factor of 20.

A factor is also a divisor, but it is considered a divisor when it separates a number into parts, not when it helps to make up a number.

Note. Unity cannot be a factor.

Factoring is the process of separating an expression into its factors.

EXAMPLE. Find the factors of $12\,a^3 b\, x^{\frac{3}{2}}$.

Solution. The prime factors of 12 are 2, 2, and 3. The factors of a^3 are a, a, and a. The factors of $x^{\frac{3}{2}}$ are x and $x^{\frac{1}{2}}$.

Therefore, $12\,a^3 b\, x^{\frac{3}{2}} = 2 \times 2 \times 3 \times a \times a \times a \times b \times x \times x^{\frac{1}{2}}$.

Hence, as a direct result of the principle that monomials are multiplied by writing the several letters in connection, and giving each an exponent equal to the sum of the exponents of that letter in the factors,

To Factor a Monomial. Separate the letters into any number of factors, so that the sum of all the exponents of each factor shall make the exponent of that factor in the given expression; also separate the numerical coefficient into its prime factors.

Exercise 43.

Separate into factors with integral exponents:

1. $12\,a^2 b^3 x$; $10\,x^3 y^2$; $15\,a b^3 c^2$; $20\,a b c^3$; $35\,x^3 y^2 z^6$; $28\,a b^2 x^2$; $36\,a b^2 x^3$.

Separate into two equal factors:

2. $16\,a^4 b^2$; $9\,x^6 y^3$; $81\,a^2 b^4 x^{4n} y^{2m}$; $169\,a^n b$.

Remove the factor $2\,a^{\frac{1}{2}} b^{\frac{1}{2}}$ from:

3. $8\,a^{\frac{3}{4}} b$; $6\,a\,b\,x$; $16\,a\,b^2 c^2$; $10\,a^{-\frac{1}{2}} b^{-1} x^3 y^3$.

Separate into three factors, also into four:

4. x; m^{2n}; a^n; $x^{\frac{1}{2}}$; $x^{\frac{2}{3}}$.

52. EXAMPLE 1. Factor $a^3 x - 3\,a^2 x^2$.

Solution. Dividing the expression by $a^2 x$, we have $a - 3\,x$. Hence, $a^3 x - 3\,a^2 x = a^2 x\,(a - 3\,x)$.

EXAMPLE 2. Factor $5\,a^2 b^2 x^3 - 15\,a\,b^2 x^3 + 20\,b^3 x^2$.

Solution. By examining the terms of the expression we find that $5\,b^2 x^2$ is a factor of every term. Dividing by this common factor the other is found. Hence, the factors are $5\,b^2 x^2$ and $a^2 x - 3\,a\,x + 4\,b$.

$\therefore\ 5\,a^2 b^2 x^3 - 15\,a\,b^2 x^3 + 20\,b^3 x^2 = 5\,b^2 x^2 (a^2 x - 3\,a\,x + 4\,b)$.

Hence,

When the Terms of a Polynomial have a Monomial Factor. Divide each term of the expression by the common factor. The divisor and quotient will be the required factors.

Exercise 44.

Factor the following:

1. $7\,n^2 + n$; $4\,a^2 b + a\,b^2 c + 3\,a\,b$; $3\,a^3 - 12\,a^2$.

2. $a\,x - b\,x + c\,x$; $39\,x^3 y^5 + 57\,x^5 y^2$.

3. $5\,x^4 + 8\,x^3 - x^3$; $72\,b^2 x^2 y^2 - 84\,b^3 x\,y^2 - 96\,a\,b\,x^2 y^3$.

4. $924\,a^2 x^n y^n z - 1178\,a\,x^n y\,z^n + 1232\,a^2 x^n y^2 z^2$.

5. $4a^2b - 60ab^3 + 20abc + 8a^2b^4x^{\frac{1}{2}} + 16aby - 36a^3bcx^{\frac{1}{2}}$.

6. $2x^{\frac{1}{2}}y - abxy + cx^2y^3$; $5x^{\frac{4}{3}} + 10x^{\frac{2}{3}} - 15x^{\frac{2}{6}}$.

7. $\frac{1}{4}ac^{-1} + \frac{1}{2}a^{-1}c^{\frac{1}{2}}b - \frac{1}{8}a^{\frac{3}{2}}b^{-\frac{1}{2}}c$; $a^{3n}x^n - a^{2n}x^{2n} + a^nx^{3n}$.

8. $a^nb^{2n}c^{3n} + a^{2n}b^{3n}c^n - a^{3n}b^nc^{2n}$.

53. In certain **Trinomials**, of the form $x^2 + ax + b$, where a and b represent any numbers, either integral, fractional, positive, or negative, it is possible to reverse the operation of Art. 25, and separate the expression into the product of two binomial factors. Evidently the first term of each factor will be the square root of x^2, or x; and to obtain the second terms of the factors, *find two numbers whose algebraic product is the last term, or* b, *and whose sum is the coefficient of* x, *or* a.

EXAMPLE 1. Factor $x^2 + 21x + 110$.

Solution. Evidently the first term of each factor will be x. The second term of the factors must be two numbers whose *product* is 110 (the third term), and whose *sum* is 21 (the coefficient of x). The only two numbers whose product is 110 and whose sum is 21 are 10 and 11. Therefore, $x^2 + 21x + 110 = (x + 10)(x + 11)$.

EXAMPLE 2. Factor $x^2 + x - 132$.

Solution. Evidently the first term of each binomial factor will be x. The second term of the two binomial factors must be two numbers whose algebraic *product* is -132 and whose *sum* is $+1$ (the coefficient of x). The only two numbers whose product is -132 and whose sum is $+1$ are $+12$ and -11. Therefore, $x^2 + x - 132 = (x + 12)(x - 11)$.

EXAMPLE 3. Factor $y^2 - 5cy - 50c^2$.

Solution. Evidently the first term of each binomial factor will be y. The second term of the two binomial factors must be two numbers whose *product* is $-50c^2$ and whose *sum* is $-5c$ (the coefficient of y). The only two numbers whose product is $-50c^2$ and whose sum is $-5c$ are $+5c$ and $-10c$. $\therefore y^2 - 5cy - 50c^2 = (y + 5c)(y - 10c)$.

EXAMPLE 4. Factor $x^2y^2 - (m - n)\,xy - mn$.

Solution. Evidently the first term of each binomial factor will be xy. The second term of the two binomial factors must be two such numbers whose *product* is $-mn$ and whose *sum* is $-(m-n)$. The only two numbers whose product is $-mn$ and whose sum is $-(m-n)$ are $+n$ and $-m$.

$\therefore\ x^2y^2 - (m-n)\,xy - mn = (xy+n)(xy-m)$. Hence,

I. If the Coefficient of the Highest Power is Unity. For the first term of each factor take the square root of one term of the trinomial; and for the second term of the factors, such numbers that their algebraic product will be another term of the trinomial, and their sum multiplied by the first term of either factor will be the remaining term of the trinomial.

Exercise 45.

Factor the following:

1. $x^2 + 19x + 88$; $x^2 - 7x + 12$; $a^8 - 20a^4 + 96$.

2. $x^2 + 35x + 216$; $b^2c^2 - 24bc + 143$.

3. $a^4b^4 + 37a^2b^2 + 300$; $a^2 + 5ab - 66b^2$.

4. $a^2b^2 - 5ab - 24$; $a^4 + 15a^2 + 44$; $a^6 + 17a^3 + 60$.

5. $a^2b^2 - 3abc - 10c^2$; $a^4 - 2a^2 - 120$; $n^2 + .8n + 1.5$.

6. $a^4 - a^2b^2 - 56b^4$; $x^2 - 9x - 90$; $x^4 + 13a^2x^2 - 300a^4$.

7. $x^2 - 15x + 44$; $m^2 + \frac{11}{4}m + \frac{15}{8}$; $x^2 - 11x - 26$.

8. $130 + 31ab + a^2b^2$; $a^2 - 20abx + 75b^2x^2$; $y^4 + 6x^2y^2 - 27x^4$; $1 + 13x + 42x^2$; $m^2 - 15am + 56a^2$.

9. $a^2 - 18axy - 243x^2y^2$; $(x+y)^2 + 5(x+y) + 4$.

10. $40a^2b^2 - 13ab + 1$; $(a-b)^2 + (a-b) - 2$.

11. $(x-y)^2-3(x-y)-15$; $x^2+54x+729$; $204-29x^2+x^4$.

12. $(a+b)^4+9(a+b)^2+8$; $x^{4n}-(b+m)x^{2n}+bm$.

13. $a^2-10ab^2c-39b^4c^2$; $x^{2n}+(a-b)x^n-ab$.

14. $x^2-9xy-70y^2$; $x^2-\frac{5}{4}x-\frac{3}{8}$; $x^{4n}-43x^{2n}+460$.

15. $x^2+\frac{1}{3}ax-\frac{1}{12}a^2$; $x^4-a^2x^2-462a^4$.

16. $x^2y^2+3xy-154$; $a^{2n}x^{4m}+14a^nx^{2m}y^n+33y^{2n}$.

17. $x^2y^2-28a^nb^nxy+187a^{2n}b^{2n}$; $x^2-\frac{1}{2}x+\frac{1}{18}$.

18. $x^{4n}y^{4n}+20a^mb^mx^{2n}y^{2n}+51a^{2m}b^{2m}$; $(x+y)^{6m}-7a^{4n}(x+y)^{3m}-98a^{8n}$; $n^4+.01n^2-.011$.

19. $x^2+\frac{4}{35}x-\frac{4}{35}$; $x^2+2xy-.21y^2$; $a^4+\frac{8}{15}a^2+\frac{1}{15}$.

By an extension of the foregoing principles we may factor some trinomials, of the form $c^2x^2+ax+bd$, where the coefficient of x^2 is a perfect square. Thus,

Example 20. Factor $4x^2+4x-3$.

Solution. The first term of each binomial factor will be the square root of $4x^2$. The second term of the two binomial factors must be two numbers whose *product* is -3 and whose *sum multiplied* by $2x$ is $+4x$. The only two numbers whose product is -3 and whose sum multiplied by $2x$ is $+4x$ are $+3$ and -1.

$$\therefore 4x^2+4x-3=(2x+3)(2x-1).$$

21. $4x^2-10x+6$; $9x^2-27x+18$; $4x^2+16ax+12a^2$.

22. $9a^2+30ab+24b^2$; $16x^2-20ax+6a^2$.

23. $25x^{10m}-\frac{5}{6}x^{5m}a^n-\frac{1}{6}a^{2n}$; $36(a-b)^{4n}+12(a-b)^{2n+2}-143(a-b)^4$.

54. We may factor some trinomials of the form $ax^2 + bx + c$. Thus,

EXAMPLE 1. Factor $8x^2 - 38x + 35$.

Solution. The first term, $8x^2$, is the product of the first terms of the binomial factors. The last term, 35, is the product of the second term of the two binomial factors. It is evident that the first term of each binomial factor might be $\pm 2x$ and $\pm 4x$, or $\pm 8x$ and $\pm x$; also the last terms of the two factors might be ± 7 and ± 5, or ± 35 and ± 1. From these we must select those that will produce the middle term, $-38x$, of the trinomial. Since $(+2x) \times (-5) + (+4x) \times (-7) = -38x$, we must take $+2x$ and $+4x$ for the first terms, and -7 and -5 for the corresponding second terms of the two binomial factors. Therefore, $8x^2 - 38x + 35 = (2x - 7)(4x - 5)$.

EXAMPLE 2. Factor $6x^4 - 5x^2y^2 - 6y^4$.

Solution. Take $+3x^2$ and $+2x^2$ for the factors of $6x^4$, and $+2y^2$ and $-3y^2$ as those of $-6y^4$. We now arrange them in binomial factors, so that the algebraic sum of their cross products shall be $-5x^2y^2$. Since $(+3x^2) \times (-3y^2) + (+2x^2) \times (+2y^2) = -5x^2y^2$, $+3x^2$ and $+2x^2$ are the first terms, and $+2y^2$ and $-3y^2$ are the corresponding second terms of the factors. $\therefore 6x^4 - 5x^2y^2 - 6y^4 = (3x^2 + 2y^2)(2x^2 - 3y^2)$. Hence,

II. **If the Coefficient of the Highest Power is not Unity** Arrange the trinomial in descending powers of a common letter. Select factors of the extreme terms and arrange them in binomial factors, so that the algebraic sum of their cross products shall be the second term of the trinomial.

Exercise 46.

Factor the following:

1. $4x^2 + 13x + 3$; $4y^2 - 4y - 3$; $12a^4 + a^2x^2 - x^4$.

2. $3 + 11x - 4x^2$; $8x^2 - 22xy - 21y^2$; $6a^2x^2 + ax - 1$.

3. $8m^6 - 19m^3 - 27$; $15a^2 - 58a + 11$; $6a^2 + 7ab - 3b^2$; $2m^2 - 13mn + 6n^2$; $3x^2 + 7x + 4$.

4. $24 + 37a - 72a^2$; $15x^2 + 224x - 15$; $4 - 5x - 6x^2$.

5. $6x^2 - 19xy + 10y^2$; $8x^2 + 14xy - 15y^2$; $15x^2 - 77x + 10$; $24x^2 + 22x - 21$; $11a^2 + 34a + 3$.

6. $18 - 33x + 5x^2$; $6x^2 - 7xy - 3y^2$; $5 + 32x - 21x^2$.

7. $24x^2 - 29xy - 4y^2$; $6x^{4n} + 19x^{2n}y^m - 7y^{2m}$.

8. $2(x+y)^2 + 5(x+y)(m+n) + 2(m+n)^2$; $2x^2 + x - 28$.

9. $2(x+y)^2 - 7(x+y)(a+b) + 3(a+b)^2$; $\frac{10}{3}x^2 + \frac{19}{6}x - \frac{5}{4}$.

10. $11(x-y)^{6n} - 23x^my^{3n}(x-y)^{3n} + 2x^{2m}y^n$; $27a^2 + 6a - 1$.

11. $8a^{2n} + 34a^n(x-y)^{mn} + 21(x-y)^{2mn}$.

55. A trinomial is a perfect square when two of its terms are positive, and the third term is twice the product of their square roots. Such trinomials are particular forms of I., and their binomial factors are equal.

EXAMPLE. Factor $4x^2 + 44xy + 121y^2$.

Solution. The first term of each binomial factor will be the square root of $4x^2$, or $2x$ and $2x$. For the second terms of the binomial factors take the square root of $121y^2$, or $11y$ and $11y$. Since the terms of the trinomial are positive the factors are $2x + 11y$ and $2x + 11y$. Therefore,

$$\begin{aligned} 4x^2 + 44xy + 121y^2 &= (2x + 11y)(2x + 11y) \\ &= (2x + 11y)^2. \end{aligned}$$ Hence,

III. If the Trinomial is a Perfect Square. Arrange the trinomial according to the powers of one letter. For one of the equal factors, find the square roots of the first and last terms, and connect these roots by the sign of the second term.

Exercise 47.

Factor the following:

1. $m^4 + n^4 + 2m^2n^2$; $m^2 + n^2 + 2mn$; $16a^4 - 120a^2bc + 225b^2c^2$; $a^6 - 4a^4 + 4a^2$.

2. $49m^6 - 140m^3n^2 + 100n^4$; $81x^4y^2 - 126a^3x^2y + 49a^6$.

3. $m^{16} - 2m^8n + n^2$; $1 - 10mn + 25m^2n^2$; $x^4 + 2x^3 + x^2$.

4. $(a+b)^2 + 16(a+b) + 64$; $m^2 + 18m + 81$.

5. $4a^4x^2 - 20a^2x^3y + 25x^4y^2$; $361a^2b^2c^2 - 76abcdmn + 4d^2m^2n^2$; $121m^2n^4 - 220mn^2p + 100p^2$.

6. $225x^4 - 30x^2y^2 + y^4$; $4a^{4n} - 4a^{2n}b^m + b^{2m}$.

7. $49m^2n^2 + \frac{28}{3}mn^3 + \frac{4}{9}n^4$; $x^2 + x + \frac{1}{4}$.

8. $\frac{4}{25}a^6 + \frac{1}{16}b^8 + \frac{1}{5}a^3b^4$; $a^6c + 6a^3b^3c + 9b^6c$.

9. $9x^2 - 3xy + \frac{1}{4}y^2$; $(m-n)^2 + 2(m-n) + 1$.

10. $(a^2 - a)^2 + 6(a^2 - a) + 9$; $4(x+y)^2 + \frac{1}{16} + x + y$.

11. $a^{\frac{2}{3}} + b^{\frac{4}{3}} - 2a^{\frac{1}{3}}b^{\frac{2}{3}}$; $m - 2m^{\frac{1}{2}} + 1$; $m^2n + mn^2 - 2m^{\frac{3}{2}}n^{\frac{3}{2}}$.

12. $x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y$; $m^2n + a^2 - 2amn^{\frac{1}{2}}$; $4x + 12nx^{\frac{1}{2}} + 9n^2$.

13. $(a+b)^{2n} - 10(a+b)^nc + 25c^2$; $\frac{9}{4}x^{5m} + \frac{16}{169} - \frac{12}{13}x^{\frac{5}{2}m}$;

56. Example. Factor $8x^3 - 27y^3$.

Solution. Evidently (Art. 34) $2x - 3y$ is a divisor of $8x^3 - 27y^3$. Dividing $8x^3$ by $2x$, we have $4x^2$, the first term of the quotient. Divide $4x^2$ by $2x$, multiply the result by $3y$, and we have $6xy$, the second term in the quotient. In like manner we find $9y^2$ for the last term in the quotient. Hence, the quotient is $4x^2 + 6xy + 9y^2$. Therefore, the factors of the binomial are $2x - 3y$ and $4x^2 + 6xy + 9y^2$.

Since the dividend is equal to the product of the divisor and quotient, $x^3 - 27y^3 = (2x - 3y)(4x^2 + 6xy + 9y^2)$. Hence, in general,

When a Binomial is the Difference of Two Equal Odd Powers of Two Numbers. Consider the binomial a dividend, and find a divisor and quotient by inspection (Art. 34). The divisor and quotient will be the required factors.

Exercise 48.

Factor the following:

1. $1 - 343x^3$; $8x^3 - 729y^6$; $216x^6 - a^3$.

2. $x^5y^5 - a^5b^5$; $x^7 - 1$; $243a^5 - b^5$; $a^3b^6 - m^3$.

3. $216a^3 - 343$; $3x - 81x^4$. **Suggestion.** Remove the monomial factor $3x$ first.

4. $a^{15} - 1024b^{10}$; $729x^3 - 1728y^3$; $x^{-5} - y^{-5}$.

5. $135x^5 - 320x^2$; $2a^6b - 64ab$; $x^{-\frac{3}{5}} - y^{-\frac{3}{5}}$.

6. $a^5b^5 - x^5y^5$; $64a^6 - 125b^3$; $x^{3n} - y^{3m}$.

57. Example. Factor $729 + a^6$.

Solution. Since 729 is the 6th power of 3, $3^2 + a^2$ (Art. 36) is a divisor of $729 + a^6$. Dividing 729 by 3^2 we have 3^4, the first term in the quotient. Divide 3^4 by 3^2, multiply the result by a^2, and we have 3^2a^2, the second term in the quotient. In like manner we find a^4 for the last term in the quotient. Hence, the quotient is $3^4 - 3^2a^2 + a^4$. Therefore, the factors of the binomial are $3^2 + a^2$ and $3^4 - 3^2a^2 + a^4$. Since the dividend is equal to the product of the divisor and quotient, $729 + a^6 = (9 + a^2)(81 - 9a^2 + a^4)$. Hence, in general,

When a Binomial is the Sum of Two Equal Odd Powers of Two Numbers. Let the student supply the method (See Art. 36).

Exercise 49.

Factor the following:

1. $32a^5 + 1$; $1 + x^5$; $x^6 + y^6$; $x^{10} + y^{10}$.
2. $a^7 + 128$; $x^6 + 729y^3$; $64x^6 + y^6$.
3. $a^5b^5 + x^{10}y^{10}$; $x^6 + 64y^6$; $1000x^3 + 1331y^3$.
4. $x^{18} + y^{18}$; $135x^5 + 320x^2$; $x^{24} + y^{24}$.
5. $x^{-5} + y^{-5}$; $x^{15} + y^{36}$; $x^5y^5 + a^5b^5$; $a^{21} + b^{54}$.
6. $a^{54} + b^{54}$; $1 + x^{12}$; $x^{3n} + y^{6m}$; $x^{-\frac{3}{5}} + y^{-\frac{3}{5}}$.
7. $a^{12n} + b^{9m}$; $32a^5b^5c^5 + 243x^5$; $1024a^5 + b^{10}$.
8. $64x^6 + 729a^6$; $\frac{1}{729}a^6 + \frac{1}{64}b^6$; $(a^2 - bc)^3 + 8b^3c^3$.

58. EXAMPLE 1. Factor $25x^2 - 64y^2$.

Solution. The square root of the first term is $5x^2$, and of the last term $8y$. Hence, since *the difference of the squares of two numbers is equal to the product of the sum and difference of the numbers* (Art. 26), $25x^2 - 64y^2 = (5x + 8y)(5x - 8y)$.

EXAMPLE 2. Factor $(5a - 4)^2 - (3a + 4x - 4)^2$.

Solution. The square root of each term of the binomial is $5a - 4$ and $3a + 4x - 4$. Adding the results for the first factor, we have $8a + 4x - 8$, or $4(2a + x - 2)$. Subtracting the second result from the first for the second factor, we have $2a - 4x$, or $2(a - 2x)$. Hence, the factors are $4(2a + x - 2)$ and $2(a - 2x)$.

Process.

$$\begin{aligned}(5a-4)^2-(3a+4x-4)^2 &= [(5a-4)+(3a+4x-4)][(5a-4)-(3a+4x-4)]\\ &= [5a-4+3a+4x-4][5a-4-3a-4x+4]\\ &= [8a+4x-8][2a-4x]\\ &= 4[2a+x-2][2(a-2x)]\\ &= 8(2a+x-2)(a-2x).\end{aligned}$$

A binomial expressing the difference between two equal even powers of two numbers may often be separated into several factors. Thus,

EXAMPLE 3. Factor $x^{16} - y^{16}$.

Solution. The square root of each term of the binomial is x^8 and y^8. Adding these results for the first factor, we have $x^8 + y^8$. Subtracting the second result from the first for the second factor, we have $x^8 - y^8$. Similarly the factors of $x^8 - y^8$ are $x^4 + y^4$ and $x^4 - y^4$. In the same way the factors of $x^4 - y^4$ are $x^2 + y^2$ and $x^2 - y^2$. Finally the factors of $x^2 - y^2$ are $x + y$ and $x - y$. Hence, the factors of the binomial are $x^8 + y^8$, $x^4 + y^4$, $x^2 + y^2$, $x + y$, and $x - y$.

Process.
$$\begin{aligned} x^{16}-y^{16} &= (x^8+y^8)(x^8-y^8) \\ &= (x^8+y^8)(x^4+y^4)(x^4-y^4) \\ &= (x^8+y^8)(x^4+y^4)(x^2+y^2)(x^2-y^2) \\ &= (x^8+y^8)(x^4+y^4)(x^2+y^2)(x+y)(x-y). \end{aligned}$$

Hence, in general,

When a Binomial is the Difference of Two Equal Even Powers of Two Numbers. Find the square root of each term of the binomial; add the results for one factor, and subtract the second result from the first for the other.

Notes: 1. The preceding method is a direct consequence of Art. 26.

2. The above method finds a practical application when it is necessary to find the difference between the squares of two numerical numbers. Thus,

$$(235)^2 - (219)^2 = (235 + 219)(235 - 219) = 454 \times 16 = 7264.$$

Exercise 50.

Factor the following:

1. $a^2x^2 - b^2y^2$; $16x^2 - 9y^2$; $25a^2x^2 - 49b^2y^4$.

2. $x^4 - y^4$; $x^4 - 81y^4$; $x^8 - y^8$; $x^{16} - y^8$.

3. $a^6b^4 - 81x^4y^6$; $1 - 100a^6b^4c^2$; $16a^{16} - 9b^6$.

4. $9a^{2n} - 4x^{4m}$; $\frac{1}{4}a^2 - \frac{1}{9}b^2$; $x^{\frac{2}{3}} - y^{\frac{2}{3}}$.

5. $x^{-4} - y^4$; $(a + b)^2 - (c + d)^2$; $(x - y)^2 - a^2$.

6. $a^2 - (x - y)^2$; $(x y + a b)^2 - 1$; $(a + b)^2 - (a - b)^2$.

7. $(a+1)^2-(b+1)^2$; $(a+1)^2-(b-1)^2$; $(753)^2-(253)^2$.

8. $(24 x + y)^2 - (23 x - y)^2$; $(1811)^2 - (689)^2$.

9. $(5 x - 2)^2 - (x - 4)^2$; $(1639)^2 - (269)^2$.

---

10. $a^{8n} - 1$; $729 x^7 y - x y^7$; $a^4 b - b^5$; $a - b$.

11. $(2 x + a - 3)^2 - (3 - 2 x)^2$; $64 x^{-6} - 729 y^{-6}$.

12. $(575)^2 - (425)^2$; $2 a - 4 x^2$; $25 a^n - 3 b^{2m}$; $x^6 - y^6$.

59. Compound expressions can often be expressed as the difference of two equal even powers of two numbers, and then factored by the foregoing principles. In many such expressions it will be necessary to rearrange, group, and factor the terms separately. Thus,

EXAMPLE 1. Factor $x^2 - y^2 + a^2 - b^2 + 2 a x - 2 b y$.

Process.

$$
\begin{aligned}
x^2 - y^2 + a^2 - b^2 + 2 a x - 2 b y &= x^2 + 2 a x + a^2 - b^2 - 2 b y - y^2 \\
&= (x^2 + 2 a x + a^2) - (b^2 + 2 b y + y^2) \\
&= (x + a)^2 - (b + y)^2 \\
&= [(x + a) + (y + b)]\,[(x + a) - (b + y)] \\
&= [a + b + x + y]\,[a - b + x - y]
\end{aligned}
$$

Explanation. Rearranging and grouping the terms, in order to form the difference of two *perfect* squares, we have the third expression. Factoring the third expression gives the fourth expression. The square root of each term of the fourth expression is $(x + a)$ and $(y + b)$. Adding these results for the first factor, we have $a + b + x + y$. Subtracting the second result from the first, we have $a - b + x - y$, for the second factor.

EXAMPLE 2 Factor $2xy + 1 - x^2 - y^2$.

Process.
$$\begin{aligned} 2xy + 1 - x^2 - y^2 &= 1 - x^2 + 2xy - y^2 \\ &= 1 - (x^2 - 2xy + y^2) \\ &= 1 - (x - y)^2 \qquad \text{Art 55.} \\ &= [1 + (x - y)][1 - (x - y)] \\ &= [1 + x - y][1 - x + y]. \end{aligned}$$

EXAMPLE 3 Factor $4a^2b^2 + 4c^2d^2 + 8abcd - (a^2 + b^2 - c^2 - d^2)^2$

Process. $4a^2b^2 + 4c^2d^2 + 8abcd - (a^2 + b^2 - c^2 - d^2)^2$
$$\begin{aligned} &= 4a^2b^2 + 8abcd + 4c^2d^2 - (a^2 + b^2 - c^2 - d^2)^2 \\ &= (2ab + 2cd)^2 - (a^2 + b^2 - c^2 - d^2)^2 \\ &= [(2ab+2cd) + (a^2+b^2-c^2-d^2)][(2ab+2cd) - (a^2+b^2-c^2-d^2)] \\ &= [2ab + 2cd + a^2 + b^2 - c^2 - d^2][2ab + 2cd - a^2 - b^2 + c^2 + d^2] \\ &= [(a^2+2ab+b^2) - (c^2-2cd+d^2)][(c^2+2cd+d^2) - (a^2-2ab+b^2)] \\ &= [(a+b)^2 - (c-d)^2][(c+d)^2 - (a-b)^2] \\ &= [(a+b) + (c-d)][(a+b) - (c-d)][(c+d) + (a-b)][(c+d) - (a-b)] \\ &= [a + b + c - d][a + b - c + d][a - b + c + d][b + c + d - a]. \end{aligned}$$

Explanation Arranging and factoring the first three terms, we have the third expression. The square root of each term of the third expression is $2ab + 2cd$ and $a^2 + b^2 - c^2 - d^2$ Adding and subtracting these results, respectively, gives the fifth expression. Rearranging ($2ab$ and $2cd$ suggest the proper arrangement) and grouping these terms, gives the sixth expression. Factoring the terms of the sixth expression, we have the seventh expression. Finally, factoring the terms of the seventh expression, we obtain the result.

Exercise 51.

Factor the following expressions:

1. $a^2 - b^2 - c^2 - 2bc$; $a^2 + y^2 - b^2 - 2ay$; $16 - a^2 - b^2 + 2ab$.
2. $25x^2 - b^2 - 6bc - 9c^2$; $x^2 + 2ax + a^2 - y^2 - 2yz - z^2$.
3. $4x^2 - 12xy + 9y^2 - 81$; $x^4 - (7x + 12)^2$; $4x^4 - 1 + 6x - 9x^2$.
4. $16x^4 - x^2 + x - \frac{1}{4}$; $9a^2 - 6a + 1 - x^2 - 8xy - 16y^2$.

5. $x^2 - y^2 + m^2 - n^2 - 2\,m\,x - 2\,n\,y$; $a^4 + b^4 - c^4 - d^4 + 2\,a^2\,b^2 - 2\,c^2\,d^2$; $(a^{2n} - b^{2n} + c^{2n})^2 - (a^{2n} + b^{2n} - c^{2n})^2$.

6. $12\,x\,y - 4\,x^2 - 9\,y^2 + z^2$; $4\,x - 1 - 4\,x^2 + 4\,a^2$.

7. $(x^2 - y^2 - z^2)^2 - 4\,y^2 z^2$; $(a^{2n} + b^{2n} - c^{4m})^2 - 4\,a^{2n} b^{2n}$.

8. $4\,x^2 - 12\,a\,x - c^2 - d^2 - 2\,c\,d + 9\,a^2$; $4\,x^2 + 9\,y^2 - 16\,z^2 - 25\,d^2 - 12\,x\,y - 40\,d\,z$; $4\,x^2 - b^2 - 2\,b\,c - c^2$.

9. $x^4 - 25\,a^6 + 8\,a^2 x^2 - 9 + 30\,a^3 + 16\,a^4$; $y^2 + 6\,b\,x - 9\,b^2 x^2 - 10\,b\,y - 1 + 25\,b^2$; $(a^{4n} - 4\,a^{2n} - 6)^2 - 36$.

10. $x^{2n} - 9\,a^2 + y^{2m} - 2\,x^n y^m - 6\,a\,b - b^2$.

11. $x^{6n} - 4\,y^{4m} + 12\,y^{2m} z + 2\,a^3 x^{3n} - 9\,z^2 + a^6$.

12. $4\,x^2 - 9\,y^2 + 16\,z^2 - 36\,n^2 - 16\,x\,z + 36\,n\,y$.

13. $a^{2n} + b^{2n} - 2\,a^n b^n - c^{2m} - k^{4m} - 2\,c^m k^{2m}$.

14. $4\,a^2 + 9\,x^2 - 16\,(y^2 + 4\,z^2) - 4\,(16\,y\,z - 3\,a\,x)$.

15. $a^2 + a\,b - 9\,c^2 + \frac{1}{4}b^2$; $a^4 - a^2 - 9 - 2\,a^2 b^2 + b^4 + 6\,a$.

60. The method for factoring a trinomial consisting of two trinomial factors depends upon the following axiom :

5. *If the same number be both added to and subtracted from another, the value of the latter will not be changed.*

EXAMPLE 1. Factor $x^4 + a^2 x^2 + a^4$.

Solution. Adding and subtracting $a^2 x^2$, we have $x^4 + 2\,a^2 x^2 + a^4 - a^2 x^2$. Factoring the first three terms of this expression, we get, $(x^2 + a^2)^2 - a^2 x^2$. Here we have the difference of two equal even powers of two expressions, and it is equal to the product of the sum and difference of their square roots. Hence, the factors are $a^2 + a\,x + x^2$ and $a^2 - a\,x + x^2$.

Process.

$$\begin{aligned} x^4+a^2x^2+a^4 &= x^4+a^2x^2+a^2x^2+a^4-a^2x^2 \\ &= x^4+2a^2x^2+a^4-a^2x^2 \\ &= (x^2+a^2)^2-a^2x^2 \\ &= (x^2+a^2+ax)(x^2+a^2-ax), \text{ or } (a^2+ax+x^2)(a^2-ax+x^2). \end{aligned}$$

EXAMPLE 2. Factor $16a^4-17a^2b^2+b^4$.

Process.
$$\begin{aligned} 16a^4-17a^2b^2+b^4 &= 16a^4-17a^2b^2+9a^2b^2+b^4-9a^2b^2 \\ &= 16a^4-8a^2b^2+b^4-9a^2b^2 \\ &= (4a^2-b^2)^2-(3ab)^2 \\ &= (4a^2+3ab-b^2)(4a^2-3ab-b^2) \\ &= (a+b)(4a-b)(a-b)(4a-b). \end{aligned}$$

Explanation. Adding and subtracting $9a^2b^2$ to the expression (to form a perfect square), arranging and factoring the terms, we have the fourth expression (the difference of two equal even powers). Factoring the fourth expression, we get the fifth expression. The factors of $4a^2+3ab-b^2$ are $a+b$ and $4a-b$. The factors of $4a^2-3ab-b^2$ are $a-b$ and $4a+b$. Hence,

When a Trinomial is the Product of Two Trinomial Factors. Make the trinomial a perfect square by adding the requisite expression. Also indicate the subtraction of the same expression. The resulting expression will be the difference of two squares. Take the sum of their square roots for one factor, and their difference for the other.

Exercise 52.

Factor the following expressions:

1. $9a^4+3a^2b^2+4b^4$; a^4+9a^2+81, $16x^4+4x^2y^2+y^4$.
2. $x^8+x^4y^4+y^8$; $81a^4-28a^2x^2+16x^4$; $m^4+m^2n^2+n^4$.
3. $4x^4+8x^2y^2+9y^4$, $a^8+a^4b^2+b^4$; $81a^4+36a^2+16$.
4. $25a^6-9a^3b^3+16b^6$; x^2+xy+y^2; $x^6+x^3y^3+y^6$.
5. $16a^8+8a^4b^3+9b^6$; $9a^4+38a^2b^4+49b^8$; p^8+p^4+1.

6. $49\,a^4 + 110\,a^2b^2 + 81\,b^4$; $9\,x^4 + 21\,x^2y^4 + 25\,y^8$.

7. $m^{4n} + m^{2n} + 1$; $x^{4n} + 16\,x^{2n} + 256$.

8. $a^2-3\,ab+b^2$; $a^{4n}-6\,a^{2n}b^{2m}+b^{4m}$; $25\,m^4-44\,m^2n^2+16\,n^4$.

61. Frequently the terms of an expression can be grouped so as to show a common factor. Thus,

EXAMPLE 1. Factor $2\,am + 3\,bm - cm - 4\,an - 6\,bn + 2\,cn$.

Process. $2\,am + 3\,bm - cm - 4\,an - 6\,bn + 2\,cn$

$$= (2\,am - 4\,an) + (3\,bm - 6\,bn) - (cm - 2\,cn)$$
$$= 2\,a\,(m - 2\,n) + 3\,b\,(m - 2\,n) - c\,(m - 2\,n)$$
$$= (m - 2\,n)\,(2\,a + 3\,b - c).$$

Explanation. Grouping the terms of the given expression in pairs; taking the common factor $2\,a$ out of the first, $3\,b$ out of the second, and c out of the third, we have the third expression. Dividing the third expression by $m - 2\,n$ (the common factor), we have $2\,a + 3\,b - c$. Hence, the factors are $m - 2\,n$ and $2\,a + 3\,b - c$.

EXAMPLE 2. Factor $12\,a^3 - 4\,a^2b - 3\,ax^2 + bx^2$.

Process.

$$12\,a^3 - 4\,a^2b - 3\,ax^2 + bx^2 = (12\,a^3 - 3\,ax^2) - (4\,a^2b - bx^2)$$
$$= 3\,a\,(4\,a^2 - x^2) - b\,(4\,a^2 - x^2)$$
$$= (4\,a^2 - x^2)\,(3\,a - b)$$
$$= (2\,a + x)\,(2\,a - x)\,(3\,a - b).$$

Explanation. Grouping the terms in pairs; taking the factor $3\,a$ out of the first, and b out of the second, we get the third expression. Dividing this by $4\,a^2 - x^2$, we have $3\,a - b$. The factors of $4\,a^2 - x^2$ are $2\,a + x$ and $2\,a - x$. Hence, the factors of the polynomial are $2\,a + x$, $2\,a - x$, and $3\,a - b$.

EXAMPLE 3. Factor $2\,mn - 2\,nx - my + xy + 2\,n^2 - ny$.

Process. $2\,mn - 2\,nx - my + xy + 2\,n^2 - ny$

$$= (2\,mn - 2\,nx + 2\,n^2) - (my - xy + ny)$$
$$= 2\,n\,(m - x + n) - y\,(m - x + n)$$
$$= (m - x + n)\,(2\,n - y).$$

EXAMPLE 4. Factor $-4ax + 4x^2 + 4ay + 4y^2 - 8xy$.

Process.
$$\begin{aligned}-4ax+4x^2+4ay+4y^2-8xy &= 4[-ax+x^2+ay+y^2-2xy]\\ &= 4[(x^2-2xy+y^2)-(ax-ay)]\\ &= 4[(x-y)(x-y)-a(x-y)]\\ &= 4(x-y)[(x-y)-a]\\ &= 4(x-y)[x-y-a].\end{aligned}$$

EXAMPLE 5. Factor $2am^2-2an^2-2am-2an+2a^2-2a^3+4a^2n$.

Solution. Removing the common factor $2a$, we have $m^2 - n^2 - m - n + a - a^2 + 2an$. Arrange the terms of this expression into the groups $m^2 - (n^2 - 2an + a^2)$, and $-(m + n - a)$. The factors of the first group are $m + n - a$ and $m - n + a$. Hence, $m^2 - n^2 - m - n + a - a^2 + 2an = m^2 - (n^2 - 2an + a^2) - (m + n - a) = (m + n - a)(m - n + a) - (m + n - a)$. Dividing this expression by the common factor, $m + n - a$, we have $m - n + a - 1$. Hence, the factors of the polynomial are $2a$, $m + n - a$, and $m - n + a - 1$. Therefore, $2am^2 - 2an^2 - 2am - 2an + 2a^2 - 2a^3 + 4a^2n = 2a(m + n - a)(m - n + a - 1)$.

Process. $2am^2 - 2an^2 - 2am - 2an + 2a^2 - 2a^3 + 4a^2n$
$$\begin{aligned}&= 2a[m^2 - n^2 - m - n + a - a^2 + 2an]\\ &= 2a[(m)^2 - (n - a)^2 - (m + n - a)]\\ &= 2a[(m + n - a)(m - n + a) - (m + n - a)]\\ &= 2a(m + n - a)[m - n + a - 1].\end{aligned}$$ Hence,

To Factor a Polynomial by Grouping its Terms. Group the terms of the polynomial so that each group shall contain the same compound factor. Factor each group and divide the result by the compound factor. The divisor and quotient will be the required factors. If the polynomial has a common simple factor, remove it first.

Note. It is immaterial what terms are taken for the different groups so that each group contains a common factor. If the groups are suitably chosen the result will always be the same, although the order of the factors may be changed. Thus, in Example 3, by a different grouping of the terms, we have

$$\begin{aligned}&2mn - 2nx - my + xy + 2n^2 - ny\\ &\quad= (2mn - my) - (2nx - xy) + (2n^2 - ny)\\ &\quad= m(2n - y) - x(2n - y) + n(2n - y)\\ &\quad= (2n - y)(m - x + n).\end{aligned}$$

Exercise 53.

Factor the following:

1. $a^2 + ab + ac + bc$; $a^2c^2 + acd - 2abc - 2bd$.

2. $am - bm - an + bn$; $4ax - ay - 4bx + by$; $a^4 + a^3 + a^2 + a$.

3. $6ax - 3bx - 6ay + 3by$; $pr + qr - ps - qs$.

4. $ax - 2bx + 2by + 4cy - 4cx - ay$.

5. $5a^2 - 5b^2 - 2a + 2b$; $6x^2 + 3xy - 2ax - ay$.

6. $2x^4 - x^3 + 4x - 2$; $a^4x^4 - a^3x^3 - a^2x^2 + 1$; $mx - 2my - nx + 2ny$; $4x - ax + 4a - a^2$.

7. $x^2 + mxy - 4xy - 4my^2$; $4a^2 + 4x^2 + 5a - 5x - 8ax$.

8. $3a^2 - 3ac - ab + bc$; $a^2x + abx + ac + aby + b^2y + bc$.

9. $5ax^2 + 3axy - 5bxy - 3by^2$; $mn + np - mp - n^2$.

10. $m^2 + np - mp - n^2$; $18y^3 - 2x^2y + 3x^3 - 27xy^2$.

11. $21a - 5c + 3ac - 2bc - 14b - 35$; $x^2 - 5xy + 6y^2 + 3x - 6y$; $x^3 - x^2 + x - 1$.

62. Example 1. Factor $x^2 + y^2 + z^2 - 2xy + 2xz - 2yz$.

Solution. The expression consists of three squares and three double products. Hence, it is the square of a trinomial which has x, y, and z for its terms. Since the sign of $2xz$ is $+$, and $2xy$ is $-$, x and z have like signs, while x and y have unlike signs. Hence, one of the two equal factors is $x - y + z$.

$\therefore\ x^2 + y^2 + z^2 - 2xy + 2xz - 2yz = (x - y + z)^2$.

Example 2. Factor $x^3 - 3x^2y + 3xy^2 - y^3$.

Solution. It is seen at a glance that the given polynomial fulfils the laws stated in Art. 29. Therefore, one of the three equal factors is $x - y$. $\therefore\ x^3 - 3x^2y + 3xy^2 - y^3 = (x - y)^3$. Hence,

When a Polynomial is a Perfect Power of an Expression. By observing the exponents, coefficients, and signs of the terms, find such expression, as raised to a given power, will produce the polynomial. This expression will be one of the equal factors.

Exercise 54.

Factor the following:

1. $a^2 + 2\,a\,b + b^2 + 2\,a\,c + 2\,b\,c + c^2$.

2. $a^2 - 2\,a\,b + b^2 - 2\,a\,c + 2\,b\,c + c^2$.

3. $a^2 + b^2 + c^2 + 2\,a\,b - 2\,a\,c - 2\,b\,c$; $16 + 32\,x + 24\,x^2 + x^4 + 8\,x^3$; $\frac{1}{9}\,a^2 - 2\,a\,b + 9\,b^2 - a + \frac{9}{4} + 9\,b$.

4. $a^5 - 15\,a^4 x + 90\,a^3 x^2 - 243\,x^5 - 270\,a^2 x^3 + 405\,a\,x^4$.

5. $a^2 - 2\,a\,b + b^2 + 2\,a\,c + c^2 - 2\,a\,d - 2\,b\,c + d^2 - 2\,c\,d + 2\,b\,d$.

6. $27\,x^3 y^3 - 108\,a^2 x^2 y^2 - 64\,a^6 + 144\,a^4\,x\,y$.

7. $m^2 - 2\,p\,x - 2\,n\,x + n^2 + p^2 - 2\,m\,n + 2\,m\,x + x^2 - 2\,m\,p + 2\,n\,p$.

Miscellaneous Exercise 55.

Factor the following:

Note. If the expression has a common simple factor, it should be first removed.

1. $10\,x^{2n} - 30\,x^n - 40$; $x^4 + x^2 + 1$; $12\,x^2 y^2 - 36\,x\,y - 48$.

2. $x^2 - .56\,x + .03$; $a^2 + \frac{85}{42}\,a + 1$; $6 + x - x^2$.

3. $3\,m^2 n^3 - 3\,m^4 n$; $16\,a^3 - 2$; $a^8 - 81$; $6\,x^5 + 48\,x^4 + 72\,x^3$.

4. $x^2 y^2 - \frac{7}{24}\,x\,y - \frac{1}{32}$; $a^6 b^2 - \frac{31}{60}\,a^3 b - \frac{1}{10}$; $9\,(a + b)^{2n} + \frac{5}{2}\,x\,y\,(a + b)^n - x^2 y^2$.

5. $a^2 - 2ax - 4a + x^2 + 4x$; $16 - 2x^2 - 4x^3 - 2x^4$; $5a^5 + a$.

6. $x^6 + \frac{19}{30}x^3 - \frac{4}{30}$; $a^{2n} + 16a^n + 63$; $\frac{16}{25}a^{4m} + (\frac{8}{5}a^p - \frac{12}{5}a^n)a^{2m} - 6a^{n+p}$.

7. $m^2 - am - nm + an$; $a^2 + 7a - 8$; $4a^2 - 4b^2 - 2a + 2b$; $49a^{2n}b^{6m} + 7(x^{\frac{1}{2}} + 3y^{\frac{1}{2}})a^n b^{3m} y^{\frac{1}{2}} + 3x^{\frac{1}{2}}y^{\frac{3}{2}}$.

8. $204 - 5a - a^2$; $x^{8n} - \frac{128}{99}x^{4n} + \frac{13}{99}$.

9. $m^2 - n^2 - mp - np$; $x^6 - x^5 + x^4 - x^3$; $a^6 - a^6 b^6$.

10. $380 - x - x^2$; $8a^{10m} - 9a^{5m} + 1$; $(x - y)^{4n} + 5(a^{2m} - b^{3m})(x - y)^{2n} - 25a^{2m}b^{3m}$.

11. $6x^2 - x - 77$; $12x^2 + 108x + 168$; $x^2 + 2xy + y^2 - 5x - 5y$; $\frac{9}{4}x^2 - \frac{1}{10}(5mn + 3y)x + \frac{1}{15}mny$.

12. $1 - \frac{1}{187}x - \frac{6}{187}x^2$; $2x^2 + 5xy - 3y^2 - 4ax + 2ay$.

13. $a^{2n} + (x + y)a^n b^n + b^{2n}xy$; $x^2 + (a + b)x - 2a^2 - ab$; $x^{-12} - y^9$; $81x^4 + 23x^2y^2 + y^4$.

14. $a^{12}b + b^{13}$; $x^3 + y^3 + 3xy(x + y)$; $x^{4n} + c(a + b)x^{2n} - ab(a - c)(b + c)$; $m^3 + n^3 + m + n$.

15. $a^2 - b^2 - c^2 + d^2 - 2(ad - bc)$; $4 + 4x + 2ay + x^2 - a^2 - y^2$.

16. $x^2 + (a + 2b)x + ab + b^2$; $x^{10n} + (a - b)x^{5n} - 2a^2 - 2ab$.

17. $250(m - n)^3 \pm 2$; $8(m + n)^3 \pm (2m - n)^3$.

18. $b^{2n}c^{2n}x^{2n} - b^n c^n - a^2 x^n - b^n c^n x^n + a^2$; $49p^4 - 15p^2q^2 + 121q^4$; $(m + n)^3 \pm (m - n)^3$.

19. $4(m - n)^3 - (m - n)$; $(m + n)^2 \pm m(m + n)$.

20. $x^2 - b(a-c)x - ac(a+b)(b+c)$; $64m^4 + 128m^2n^2 + 81n^4$; $25x^4 + 31x^2y^2 + 16y^4$.

21. $6x^3 + 13x^2y + 6xy^2$; $6x^2y^3 - xy^4 - 12y^5$.

22. $x^{2n} - (3bc + ac + ab)x^n + 3abc(b+c)$.

23. $m^3 + 4mn^2 \pm 8n^3 \pm 2m^2n$; $(m+3n)^2 - 9(m-p)^2$.

24. $x^{20n} + (a+b-c)x^{10n} - ac - bc$; $(x+y)^2 - x - y - 6$; $25x^4 + 24x^2y^4 + 16y^8$.

25. $9x^4y^4 - 3x^3y^5 - 6x^2y^6$; $m^2 - mn - 6n^2 \pm 4m \mp 12n$.

26. $m^{\frac{2}{3}}n^{\frac{2}{5}}(ab-z)^3 - m^{\frac{2}{3}}n^{\frac{2}{5}}(xy+2z)^3$; $81a^{2n} - 199a^nb^m + 121b^{2m}$; $81a^{4n} - 99a^{2n}b^{4m} + 25b^{8m}$.

27. $18x^2 - 24xy + 8y^2 \pm 36x \mp 24y$; $2m^2 + 2mn - 12n^2 - 12am - 3an$; $a^8x^6 \pm 64a^2n^6$.

28. $x^2 + 3xy - 28y^2 + 28y - 4x$; $2y - 6ay + 4bx + 6ax - 2x - 4by$.

29. $m^4n - m^2n^3 - m^3n^2 + mn^4$; $m^4 - (m+a)^4$.

30. $15x^2 - 16y^2 - 15ax - 8xy + 20ay$; $(a-b)(a^2-c^2) - (a-c)(a^2-b^2)$.

31. $c^5d^3 - c^2 - a^2c^3d^3 + a^2$; $m^3n^3 \pm 512$; $24m^2n^2 - 30mn^3 - 36n^4$; $ax^2 - 3bxy - axy + 3by^2$.

32. $m^2 - mn - 6n^2 + 16m - 32n$; $4x^5 + 4x^3 - x^2 - x^4$.

33. $4m^3 - 4n^3 - 3n(n^2 - m^2) + 2m(n-m)^2$.

34. $9m^9 \pm 9a^2m^7$; $x^2 - 16y^2 + x \pm 4y$; $(x - 2xy)^2 - (x - 2xy) - 6$. **Query.** How many factors in the first part?

35. $64(4x+y)^2-49(2x-3y)^2$; $(m^4-m^2-5)^2-25$.

36. $m^6+m^5n+mn^5+m^2n^4+m^4n^2+m^3n^3$; $(x-y)^2-1+xy(x-y+1)$; $(x^2+4)^2-16x^2$.

37. $(m^2+3m)^2-14(m^2+3m)+40$; $(mn-n)^2-mn(mn-n-3)-9$.

38. $x^{2n}-x^n-\frac{7}{4}+x^{-n}+x^{-2n}$; $x^{-\frac{2}{3}}-y^{-\frac{2}{3}}$.

39. $14a^2x^3-35a^3x^2+14a^4x$; $x^{-\frac{4}{5}}-y^{-\frac{4}{5}}$.

40. $12x^5-8x^3y^2+21x^2y$; $64x^{\frac{4}{2}}\pm 27x^{\frac{1}{2}}$.

Separate into four factors:

41. $(x-2y)x^3-(y-2x)y^3$; $(x^{2m}+6x^m+7)^2-(x^m+3)^2$.

42. $4a^2(x^3+18ab^2)-(32a^5+9b^2x^3)$; $16m^2n^2-(m^2+4n^2-p^2)^2$; $(a^{4m}-2a^{2m}b^{2n}-b^{4n})^2-4a^{4m}b^{4n}$.

43. $x^9+x^3y^6-8x^6y^3-8y^9$; $x^{9m}+x^{6m}+64x^{3m}+64$.

44. $m^4-2(n^2+p^2)m^2+(n^2-p^2)^2$.

Separate into five factors:

45. $m^7-m^5n^2+2m^4n^3-m^3n^4$; $6m^4n^2+m^3n-6m^3n^3-m^2n^2$; $(x^{2m}+y^{2n}-20)^2-(x^{2m}-y^{2n}+12)^2$.

46. $x^{7m}+x^{4m}-16x^{3m}-16$; $16x^{7m}-81x^{3m}-16x^{4m}+81$.

Separate into seven factors:

47. $a^{12m}-a^{8m}b^{4n}-a^{4m}b^{8n}+b^{12n}$.

CHAPTER XII.

HIGHEST COMMON FACTOR.

63. THE product of *any of the factors* of a number is a factor of the given number.

Thus, since $30 = 2 \times 3 \times 5$, 6, 10, and 15 are factors of 30.

The product of the *common factors* of two or more numbers must be a factor of each.

Thus, since $42 = 2 \times 3 \times 7$, and $66 = 2 \times 3 \times 11$, 2×3, or 6, is a factor of 42 and 66.

The product of the *highest powers* of all the factors which are common to two or more numbers must be the *greatest common factor* of the given numbers.

Thus, since $24 = 2^3 \times 3$ and $36 = 2^2 \times 3^2$, $2^2 \times 3$, or 12, is the greatest common factor of 24 and 36.

The **Highest Common Factor** (H. C. F.) of two or more algebraic expressions is the expression of **highest degree** which will divide each of them exactly.

Thus, $3x^2y^2$ is the H. C. F. of $3x^3y^3$, $6x^2y^2$, and $15x^4y^5z$.

Note 1. Two or more expressions are said to be *prime to each other* when they have no common factor. Thus, $5a^2$ and $9b$ are prime to each other.

EXAMPLE 1. Find the H. C. F. of $24a^3b^3c^3$, $60a^3b^3c^2y^2$, $48a^3b^2c^3$, and $36a^2b^2c^5x^3$.

Process.
$$\begin{aligned} 24\,a^3b^3c^3 &= 2^3 \times 3 \times a^3 \times b^3 \times c^3;\\ 60\,a^3b^3c^2y^2 &= 2^2 \times 3 \times 5 \times a^3 \times b^3 \times c^2 \times y^2;\\ 48\,a^3b^2c^3 &= 2^4 \times 3 \times a^3 \times b^2 \times c^3;\\ 36\,a^2b^2c^5x^3 &= 2^2 \times 3^2 \times a^2 \times b^2 \times c^5 \times x^3. \end{aligned}$$

$\therefore$ the H. C. F. $= 2^2 \times 3 \times a^2 \times b^2 \times c^2 = 12\,a^2b^2c^2$.

Explanation. Factoring each expression, it is seen that the only factors *common* to each are 2^2, 3, a^2, b^2, and c^2. Hence, all of these expressions can be divided by any of these factors, or by their product, and by no other expression.

EXAMPLE 2. Find the H. C. F. of $2\,x^3 - 2\,x\,y^2$, $4\,x^5 - 4\,x\,y^4$, and $2\,x^4 - 2\,x^2y^2 + 2\,x^3y - 2\,x\,y^3$.

Process.
$$\begin{aligned} 2\,x^3 - 2\,x\,y^2 &= 2\,x\,(x+y)\,(x-y);\\ 4\,x^5 - 4\,x\,y^4 &= 2^2\,x\,(x+y)\,(x-y)\,(x^2+y^2);\\ 2\,x^4 - 2\,x^2y^2 + 2\,x^3y - 2\,x\,y^3 &= 2\,x\,(x+y)^2\,(x-y); \end{aligned}$$

$\therefore$ the H. C. F. $= 2\,x\,(x+y)\,(x-y) = 2\,x\,(x^2 - y^2)$.

Explanation. Factoring each expression it is seen that the only factors common to each are 2, x, $x + y$, and $x - y$. Hence, all of these expressions can be divided by any of these factors, or by their product, and by no other expression.

Note 2. If the expressions contain different powers of the *same factor*, the H. C. F. must contain the highest power of the factor which is common to all of the given expressions.

EXAMPLE 3. Find the H. C. F. of $8\,a^5x^2 + 16\,a^4x^3 + 8\,a^3x^4$, $2\,a^4x^2 - 4\,a^5x - 6\,a^6$, $6\,(a^2 + a\,x)^2$, and $24\,(a^2x + a\,x^2)^3$.

Process.

$$\begin{aligned} 8a^5x^2 + 16\,a^4x^3 + 8\,a^3x^4 &= 2^3 \times a^3 \times x^2\,(a+x)^2;\\ 2\,a^4x^2 - 4\,a^5x - 6\,a^6 &= -2 \times a^4 \times (a+x)\,(3\,a - x);\\ 6\,(a^2 + a\,x)^2 &= 2 \times 3 \times a^2 \times (a+x)^2;\\ 24\,(a^2x + a\,x^2)^3 &= 2^3 \times 3 \times a^3 \times x^3\,(a+x)^3. \end{aligned}$$

The common factors are 2, a^2, and $a + x$.

$\therefore$ the H. C. F. $= 2\,a^2\,(a + x)$. Hence,

To Find the H. C. F. of Two or more Expressions that can be Factored by Inspection. Separate the expressions into their factors. Take the product of the common factors, giving to each factor the highest power which is common to all the given expressions.

Exercise 56.

Find the H.C.F. of:

4. $12\,a^2b^3x^2$ and $18\,a^2b\,x^3$; $6\,a^2xy$, $8\,a\,x^3y$, and $9\,a^3xy^4$.

5. $15\,a^3x^3y^4$, $9\,a^3x^2y^3$, and $21\,a^2x^3y^2$.

6. $12\,x^3y^2z^2$, $18\,x^4y^3z^5$, and $36\,x^2y^4z^3$.

7. $20\,c^3x^{\frac{3}{2}}y^3$, $8\,a^2x^2y^{\frac{3}{4}}$, and $12\,a^2x^{\frac{2}{3}}y^{\frac{3}{2}}$.

8. $a^2b\,x + a\,b^2x$ and $a^2b - b^3$.

9. $x^2y^2 - z^2$ and $a\,x^2y - b\,x\,y + a\,x\,z - b\,z$.

10. $3\,x^4+8\,x^3+4\,x^2$, $3\,x^5+11x^4+6\,x^3$, and $3\,x^4-16\,x^3-12x^2$.

11. $3\,a^2x^2y - 3\,a^2x\,y - 36\,a^2y$ and $3\,a^2x^3 - 48\,a^2x - 3\,a^2x^2 + 48\,a^2$.

12. $x^2 + x$, $(x+1)^2$, and $x^3 + 1$; $x^{2n} + x^n - 30$ and $x^{2n} - x^n - 42$; $x^3 + 27$, $x^2 - 9$, and $2\,x^2 + 5\,x - 3$.

13. $x^3 - x^2y$, $x^3 - xy^2$, and $x^4 - x\,y^3$.

14. $x^4 + x^2y^2 + y^4$ and $x^3 - 2\,x^2y + 2\,x\,y^2 - y^3$.

15. $12\,(a-b)^4$, $8\,(a^2-b^2)^2$, and $20\,(a^4-b^4)$.

16. $8\,x\,z\,(x-y)\,(x-z)$ and $12\,y\,z\,(y-x)\,(y-z)$.

17. $4\,x^2 - 12\,x + 9$, $4\,x^2 - 9$, and $4\,a^2b\,x - 6\,a^2b$.

18. $x^3 - 27\,y^3$, $x^2 - 6\,x\,y + 9\,y^2$, and $2\,x^2 - x\,y - 15\,y^2$.

19. $x^4 - y^4$, $(x^2 - y^2)^2$, and $a x^2 - 7 a x y + 6 a y^2$.

20. $m x^3 - m x$, $2 x^3 + 18 x^2 - 20 x$, and $4 a^2 x^6 - 4 a^2 x$.

21. $24 m n - 6 m + 16 p n - 4$, $64 n^2 - 4$, and $16 n^2 - 8 n + 1$.

22. $x^2 + 4 x + 4$, $x^3 + 8$, and $4 x^2 + 2 x - 12$.

23. $16 x^3 - 432$, $x^2 - 6 x + 9$, and $5 x^2 - 13 x - 6$.

24. $m^2 - n^2$, $m n - n^2 + m p - n p$, and $m^3 - m^2 n + m n^2 - n^3$.

25. $6 x^5 - 96 x$, $m x^3 y - 8 m y$, and $15 p x^2 - 60 p$.

26. $x^{6n} - 11 x^{3n} + 30$, $x^{6n} - 13 x^{3n} + 42$, and $x^{6n} + x^{3n} - 42$.

27. $x^{3n} - 125$, $x^{2n} - 10 x^n + 25$, and $2 x^{2n} - 11 x^n + 5$.

28. $8 x^{3n} - 125$, $4 x^{2n} - 25$, and $4 x^{2n} - 20 x^n + 25$.

64. If the expressions cannot readily be factored by inspection, we adopt a method analogous to that used in arithmetic for the greatest common divisor of two or more numbers. The method depends on two principles :

1. *A factor of any expression is a factor of any multiple of that expression.*

Thus, 4 is contained in 16, 4 times; it is evident that it is contained in 5 times 16, or 80, 5 times 4, or 20 times. In general,

Since a factor is a divisor, if a represent a factor of any expression, m, so that a is contained in m, b times, it is evident that it is contained in $r m$, r times b, or $r b$ times.

2. *A common factor of any two expressions is a factor of their sum and their difference, and also the sum and the difference of any multiple of them.*

Thus, 4 is contained in 36, 9 times, and in 16, 4 times. Hence, it is contained in $36+16$, $9+4$, or 13 times, and in $36-16$, $9-4$, or 5 times. Again, 4 is contained in 5 times 36, 5 times 9, or 45 times; also, 4 is contained in 10 times 16, 10 times 4, or 40 times. Hence, it is contained in $180+160$, $45+40$, or 85 times, and in $180-160$, $45-40$, or 5 times. In general,

Let a be a factor of m and n, so that a is contained in m, b times, and in n, c times. Then $(m+n) \div a = b+c$; also, $(m-n) \div a = b-c$. Again, since a is contained in m, b times, it is evident that it is contained in rm, r times b, or rb times; also, since a is contained in n, c times, it is contained in sn, s times c, or sc times. Hence, $rm \div a = rb$, and $sn \div a = sc$. Adding these equations, we have $(rm+sn) \div a = rb+sc$; subtracting the second equation from the first, we have $(rm-sn) \div a = rb-sc$. The last two equations may be written $(rm \pm sn) \div a = rb \pm sc$. Therefore, $rm \pm sn$ contains the factor a.

EXAMPLE 1. Find the H.C.F. of $4x^3-3x^2-24x-9$ and $8x^3-2x^2-53x-39$.

Solution. The H.C.F. cannot be of higher degree than the first expression. If the first expression divides $8x^3-2x^2-53x-39$, it is the H.C.F. By trial, we find a remainder, $4x^2-5x-21$. The H.C.F. of the given expressions is also a divisor of $4x^2-5x-21$, because $4x^2-5x-21$ is the difference between $8x^3-2x^2-53x-39$ and 2 times $4x^3-3x^2-24x-9$ (Principle 2). Therefore, the H.C.F. cannot be of higher degree than $4x^2-5x-21$. If $4x^2-5x-21$ exactly divides $4x^3-3x^2-24x-9$, it will be the H.C.F. By trial, we find a remainder, $2x^2-3x-9$. The H.C.F. of $4x^2-5x-21$ and $4x^3-3x^2-24x-9$ is also a divisor of $2x^2-3x-9$, because $2x^2-3x-9$ is the difference between $4x^3-3x^2-24x-9$ and $x+1$ times $4x^2-5x-21$ (Principle 2). Therefore, the H.C.F. cannot be of higher degree than $2x^2-3x-9$. If $2x^2-3x-9$ exactly divides $4x^2-5x-21$, it will be the H.C.F. By trial, we find a remainder, $x-3$. The H.C.F. of $2x^2-3x-9$ and $4x^2-5x-21$ is also a divisor of $x-3$, because $x-3$ is the difference between $4x^2-5x-21$ and 2 times $2x^2-3x-9$ (Principle 2). Therefore, the H.C.F. cannot be of higher degree than $x-3$. If $x-3$ exactly divides $2x^2-3x-9$, it will be

the H. C. F. By trial, we find that $x - 3$ is an exact divisor of $2x^2 - 3x - 9$. Therefore $x - 3$, or $3 - x$ is the H. C. F.

Process.

$$
\begin{array}{lr}
 & 4x^3 - 3x^2 - 24x - 9\,)\,8x^3 - 2x^2 - 53x - 39\,(2 \\
\text{2 times the divisor,} & 8x^3 - 6x^2 - 48x - 18 \\
\text{First remainder,} & 4x^2 - 5x - 21 \\
 & 4x^2 - 5x - 21\,)\,4x^3 - 3x^2 - 24x - 9\,(x + 1 \\
x \text{ times second divisor,} & 4x^3 - 5x^2 - 21x \\
\text{Second remainder,} & 2x^2 - 3x - 9 \\
 & 2x^2 - 3x - 9\,)\,4x^2 - 5x - 21\,(2 \\
\text{2 times third divisor,} & 4x^2 - 6x - 18 \\
\text{Third remainder,} & x - 3 \\
 & x - 3\,)\,2x^2 - 3x - 9\,(2x + 3 \\
2x \text{ times fourth divisor,} & 2x^2 - 6x \\
\text{Fourth remainder,} & 3x - 9 \\
\text{3 times fourth divisor,} & 3x - 9
\end{array}
$$

Therefore, the H. C. F. $= x - 3$, or $3 - x$.

Note 1. The signs of all the terms of the remainder may be changed; for if an expression A is divisible by $-B$, it is divisible by $+B$. Hence, in the above example, the H. C. F. is $x - 3$, or $3 - x$.

EXAMPLE 2. Find the H.C.F. of $4x^3 - x^2y - xy^2 - 5y^3$ and $7x^3 + 4x^2y + 4xy^2 - 3y^3$.

Process.

$$
\begin{array}{lr}
 & 4x^3 - x^2y - xy^2 - 5y^3\,)\,7x^3 + 4x^2y + 4xy^2 - 3y^3\,(7 \\
\text{4 times first dividend,} & 28x^3 + 16x^2y + 16xy^2 - 12y^3 \\
\text{7 times first divisor,} & 28x^3 - 7x^2y - 7xy^2 - 35y^3 \\
\text{First remainder,} & 23x^2y + 23xy^2 + 23y^3 = 23y(x^2 + xy + y^2) \\
 & x^2 + xy + y^2\,)\,4x^3 - x^2y - xy^2 - 5y^3\,(4x - 5y \\
4x \text{ times second divisor,} & 4x^3 + 4x^2y + 4xy^2 \\
\text{Second remainder,} & -5x^2y - 5xy^2 - 5y^3 \\
-5y \text{ times second divisor,} & -5x^2y - 5xy^2 - 5y^3
\end{array}
$$

Therefore, the H. C. F. $= x^2 + xy + y^2$.

Explanation. Arrange according to descending powers of x, take for the divisor the expression whose highest power has the smaller coefficient, and multiply the dividend by 4 (to avoid fractions). Since 4 is not a factor of $4x^3 - x^2y - xy^2 - 5y^3$, the H. C. F. of the

given expressions is the H. C. F. of $4x^3 - x^2y - xy^2 - 5y^3$ and $28x^3 + 16x^2y + 16xy^2 - 12y^3$ (Principle 2). Since $23y(x^2 + xy + y^2)$ is the difference between 4 times the dividend and 7 times the divisor, the H. C. F. of the given expressions is a divisor of it (Principle 2). Therefore, the H. C. F. cannot be of higher degree than $23y(x^2 + xy + y^2)$ If the first remainder exactly divides the first divisor, it will be the H. C. F. Since $23y$ is not a factor of the first divisor, it can be rejected. Therefore, $x^2 + xy + y^2$ is the H. C. F.

This method is used only to determine the **compound** factor of the H. C. F. If the given expressions have **simple factors**, they must first be separated from them, and the H. C. F. of these must be reserved and multiplied into the compound factor obtained. Thus,

EXAMPLE 3. Find the H. C. F. of $54x^6y + 60x^2y^5 - 18x^3y^4 - 132x^4y^3$ and $18x^6y^2 - 50x^2y^6 + 2x^4y^4 - 12x^5y^3$.

Solution. Removing the simple factors $6x^2y$ and $2x^2y^2$, and *reserving their highest common factor*, $2x^2y$, *as forming a part of the* H. C. F., we are to determine the compound factor of $9x^4 - 22x^2y^2 - 3xy^3 + 10y^4 = A$ and $9x^4 - 6x^3y + x^2y^2 - 25y^4 = B$. If A exactly divides B, it is the H. C. F. of A and B. By trial, we find the remainder $-y(6x^3 - 23x^2y - 3xy^2 + 35y^3)$. The H. C. F. of A and B is also a divisor of this remainder, because the remainder is the difference between B and 1 times A (Principle 2). Reject $-y$ from this remainder, since it is not a common factor of A and B, and represent the result by D. The H. C. F. of D and $2A$ (a multiple of A) is the H. C. F. of A and B (Principle 2). This cannot be of higher degree than D; and if D exactly divides $2A$, it is the H. C F. By trial, we find a remainder, $153y^2(3x^2 - xy - 5y^2)$. The H. C. F. of D and $2A$ is also a divisor of this remainder. Reject $153y^2$, and represent the result by E. The H. C. F. of E and D is the H. C. F. of D and $2A$; and if E exactly divides D, it is the H. C. F. By trial, we find that E is an exact divisor of D. Therefore, E is the H. C. F. of A and B. Hence, the H. C. F. of the given expressions is $2x^2y(3x^2 - xy - 5y^2)$.

Process.

$$9x^4-22x^2y^2-3xy^3+10y^4 = A\,)\,9x^4-6x^3y+x^2y^2 \qquad -25y^4 = B\,(\,1$$

1 times the first divisor, $\underline{9x^4 \qquad -22x^2y^2-3xy^3+10y^4}$

$$-6x^3y+23x^2y^2+3xy^3-35y^4$$
$$= -y(6x^3-23x^2y-3xy^2+35y^3)$$

$$(\,3x+23y$$

$$6x^3-23x^2y-3xy^2+35y^3 = D\,)\,18x^4 \qquad -44x^2y^2-\ 6xy^3+\ 20y^4 = 2A$$

$3x$ times second divisor, $\underline{18x^4-69x^3y-\ 9x^2y^2+105xy^3}$

Second remainder, $69x^3y-\ 35x^2y^2-111xy^3+\ 20y^4$

2 times second remainder, $138x^3y-\ 70x^2y^2-222xy^3+\ 40y^4$

$23y$ times second divisor, $\underline{138x^3y-529x^2y^2-\ 69xy^3+805y^4}$

Third remainder, $459x^2y^2-153xy^3-765y^4$

$$= 153y^2(3x^2-xy-5y^2)$$

$$3x^2-xy-5y^2 = E\,)\,6x^3-23x^2y-\ 3xy^2+35y^3 = D\,(\,2x-7y$$

$2x$ times third divisor, $\underline{6x^3-\ 2x^2y-10xy^2}$

Fourth remainder, $-21x^2y+\ 7xy^2+35y^3$

$-7y$ times third divisor, $\underline{-21x^2y+\ 7xy^2+35y^3}$

Therefore, H. C. F. $= 2x^2y\,(3x^2-xy-5y^2)$.

EXAMPLE 4. Find the H. C. F. of $90x^5y^2-200x^2y^5-10x^3y^4$ and $144x^4y-64xy^4-16x^2y^3-144x^3y^2$.

Removing the simple factors $10x^2y^2$ and $16xy$, and *reserving their highest common factor*, 2 x y, *as forming part of the* H. C. F., arranging according to descending powers of x, we have

Process. $9x^3-xy^2-20y^3\,)\,9x^3-9x^2y-\ xy^2-\ 4y^3\,(\,1$

1 times the first divisor, $\underline{9x^3 \qquad -\ xy^2-20y^3}$

First remainder, $-9x^2y \qquad +16y^3 = -y\,(9x^2-16y^2)$

$$9x^2-16y^2\,)\,9x^3-\ xy^2-20y^3\,(\,x$$

x times second divisor, $\underline{9x^3-16xy^2}$

Second remainder, $15xy^2-20y^3 = 5y^2(3x-4y)$

$$3x-4y\,)\,9x^2-16y^2 = (3x+4y)(3x-4y)\,(\,3x+4y$$

$3x+4y$ times third divisor, $\underline{(3x+4y)(3x-4y)}$

$\therefore$ the H.C.F. $= 2xy\,(3x-4y)$. Hence, in general,

To Find the H.C.F. of Two Polynomials that cannot readily be Factored by Inspection. If the given expressions have simple factors, remove them and arrange the resulting expressions according to powers of a common letter. Take that expression which is of lower degree for the first divisor; or, if both are of the same degree, that whose first term has the smaller coefficient. If there is a remainder, divide the first divisor by it, and continue to divide the last divisor by the last remainder, until there is no remainder. The last divisor will be their highest common factor. The highest common factor of the simple factors multiplied by the last divisor will give the H. C. F. sought.

Notes: 2. If the first term of the dividend or of any remainder is not exactly divisible by the first term of the divisor, that dividend or remainder must be *multiplied* by such an expression as will make the first term thus divisible.

3. Observe that we may multiply or divide either of the polynomials, or any of the remainders which occur in the course of the work, by any factor which does not divide both of the polynomials, as such a factor can evidently form no part of the H. C. F.

Exercise 57.

Find the highest common factor of:

1. $x^3 + 2x^2 - 13x + 10$ and $x^3 + x^2 - 10x + 8$.

2. $x^4 - 2x^2 + 1$ and $x^4 - 4x^3 + 6x^2 - 4x + 1$.

3. $x^3 - x^2 - 5x - 3$ and $x^3 - 4x^2 - 11x - 6$.

4. $x^4 - 9x^3 + 29x^2 - 39x + 18$ and $4x^3 - 27x^2 + 56x - 33$.

5. $x^3 - 5ax^2 + 4a^2x$ and $x^4 - ax^3 + 3a^2x^2 - 3a^3x$.

6. $2y^3 - 10xy^2 + 8x^2y$ and $9x^4 - 3xy^3 + 3x^2y^2 - 9x^3y$.

7. $2x^5 - 11x^2 - 9$ and $4x^5 + 11x^4 + 81$.

8. $18x^2 + 3x - 6$ and $18x^3 + 96x^2 + 104x + 32$.

9. $15\,m^2\,n^3 - 20\,m^2\,n^2 - 65\,m^2 n - 30\,m^2$ and $2\,a\,m\,n^3 + 20\,a\,m\,n^2 - 16\,a\,m\,n - 16\,a\,m$.

10. $36\,m^6 + 9\,m^3 - 27m^4 - 18\,m^5$ and $27m^5 n^2 - 9\,m^3 n^2 - 18\,m^4\,n^2$.

11. $3\,x^2 - 3\,x^2 y + x\,y^2 - y^3$ and $4\,x^2 y - 5\,x\,y^2 + y^2$.

12. $m\,n\,x^5 - 82\,m\,n\,x - 3\,m\,n$ and $m^5\,n^2\,x^5 + 28\,m^5\,n^2 x^2 - 9\,m^5 n^2$.

13. $x^3 - 4\,x^2 + 2\,x + 3$ and $2\,x^4 - 9\,x^3 + 12\,x^2 - 7$.

14. $16\,x^2 - 44\,x\,y + 10\,y^2$ and $6\,x^4 - 29\,x^3\,y + 43\,x^2\,y^2 - 20\,x\,y^3$.

15. $2\,m^5 n - 10\,m^3\,n\,y^2 + 18\,m^2 n\,y^3 + 224\,m\,n\,y^4 + 294\,n\,y^5$ and $4\,m^4\,y - 20\,m^2\,n^2\,y - 48\,m\,n^3\,y + 112\,n^4\,y$.

16. $2\,m^n x^{4n} - 2\,m^n x^{3n} - 4\,m^n x^{2n} + 4\,m^n x^n$ and $6\,m^n x^{5n} - 18\,m^n x^{4n} + 12\,m^n x^{3n} + 6\,m^n x^{2n} - 6\,m^n x^n$.

Query. How many factors in this result?

65. To Prove the Method for Finding the H. C. F. of any Two Algebraic Expressions. Let A and B represent the expressions, the degree of A being either equal to or higher than that of B. Divide A by B, and let the quotient be m and the remainder D; divide B by D, and let the quotient be n and the remainder E; divide D by E, and let the quotient be r and the remainder *zero;* that is, E is supposed to be exactly contained in D.

Process.

$$
\begin{array}{llll}
B\,)\,A\,(\,m & & & \\
\underline{m\,B\quad} & & & \\
\quad D\,)\,B\,(\,n & & & \\
\quad \underline{n\,D\quad} & & & \\
\qquad E\,)\,D\,(\,r & & & \\
\qquad \underline{r\,E\quad} & & & \\
\qquad\quad 0 & & &
\end{array}
$$

We will first prove that E is a *common* factor of A and B.

From the nature of subtraction, the minuend is equal to the subtrahend and remainder. Hence, $A = m\,B + D$, $B = n\,D + E$, and

$D = rE$. Since the division has terminated, E is a divisor of D. E is also a divisor of nD (Principle 1) and of $nD + E$, or B (Principle 2). Hence, E is a divisor of mB (Principle 1), and of $mB + D$, or A (Principle 2). Therefore, E **is a common factor of** A **and** B.

We must now show that E is the *highest* common factor.

Every divisor of A and B is also a divisor of mB (Principle 1), and of $A - mB$, or D (Principle 2). Therefore, every divisor of A and B is a divisor of nD (Principle 1), and of $B - nD$, or E (Principle 2). But no divisor of E can be of *higher* degree than E itself. Therefore, E **is the highest common factor of** A **and** B.

66. Let A, B, D, E, etc. represent any polynomials. Let m represent the H. C. F. of A and B, n the H. C. F. of m and D, and p the H. C. F. of n and E, etc. Evidently m is the product of *all* the factors common to A and B; also, n is the product of *all* the factors common to m and D, and p is the product of *all* the factors common to n and E, or p is the product of *all* the factors common to A, B, D, and E, etc., which is their H. C. F. Hence, in general,

To Find the H. C. F. of Several Polynomials. Find the H. C. F. of two of them; then of this result and one of the remaining polynomials; and so on. The last result found will be the H. C. F. of the given polynomials.

Exercise 58.

Find the H. C. F. of:

1. $10x^5 + 10x^3y^2 + 20x^4y$, $12xy^3 + 4x^3y + 12x^2y^2 + 4y^4$, and $2x^3 + 2y^3$.

2. $x^4 + x^3 - 8x^2 - 9x - 9$, $2 + x + x^5 + x^3 + 2x^4 + 2x^2$, and $3 + 3x^2 + x^5 + x^3 + 3x^4$.

3. $m^n x^3 + 2m^n x^2 + m^n x + 2m^n$, $2x^4 + 6x^3 + 4x^2$, $3x^3 + 9x^2 + 9x + 6$, $3x^5 - 12x^3 - 3x^2 - 6x$, and $3x^3 + 2 + 5x + 8x^2$.

4. $2x^3 - 5x + 6 - 3x^2$, $3x^2 + 2x^3 - 8x - 12$, and $2x^3 - x^2 - 12x - 9$.

5. $3x^{3n} - 33x^{2n} + 96x^n - 84$, $68x^{2n} - 92x^{3n} - 24x^n + 32x^{4n}$, $x^{3n} + 11x^n - 6 - 6x^{2n}$, $50x^n + 20x^{3n} - 60x^{2n} - 20$, $5x^{3n} - 10x^{2n} + 7x^n - 14$, and $3x^{6n} - 35x^{5n} - 162x^{4n} - 372x^{3n} + 424x^{2n} - 192x^n$.

6. $9x^{2n} + 4x^n + 2x^{3n} - 15$, $48x^{3n} + 30 - 348x^n - 24x^{2n}$, $8x^{2n} + 4x^{3n} + 3x^n + 20$, and $2x^{2n} + 12x^{3n} - 94x^n - 60$.

7. $3x^3 - 2xy^2 - 5x^2y$, $5xy^3 - 6y^4 - 3x^2y^2 - x^3y + x^4$, $9x^3 - 8x^2y - 20xy^2$, $3xy^2 - 7x^2y - 2y^3 + 3x^3$, $10y^4 - x^2y^2 - 5x^3y + 3x^4 - 7xy^3$, and $x^4 - x^3y - x^2y^2 - xy^3 - 2y^4$.

Miscellaneous Exercise 59.

Note. When possible the student should separate the given expressions into their factors by inspection.

Find the H.C.F. of:

1. $x^5 - xy^2$ and $x^3 + x^2y + xy + y^2$.

2. $x^2 - y^2$, $(x + y)^2$, $x^3 - x^2y$, and $2x^2y - 2xy$.

3. $2x^2 - x - 1$, $xy - y$, $x^4y - xy$, and $3x^2 - x - 2$.

4. $x^6 - 6x + 5$, $2x^3 + 5 - 8x + x^2$, $x^6 + x^5 - 11x + 9$, and $42x^2 + 30 - 72x$.

5. $x^2 - 18x + 45$, $2x^2 - 7x + 3$, $x^2 - 9$, and $3x^2 - 7x - 6$.

6. $6x^{4n} - 3x^{3n} - x^{2n} - x^n - 1$ and $3x^{4n} - 3x^{3n} - 2x^{2n} - x^n - 1$.

7. $x^3 - y^3$, $x^4 + x^2y^2 + y^4$, $x^6 - y^6$, $x^2 + xy + y^2$, $x^4 + x^3y - xy^3 - y^4$, and $x^3y + x^2y^2 + xy^3$.

8. $2ax^2 + 2a + 4ax$, $x^3 + 2x^2 + 2x + 1$, $7b + 14bx + 7bx^3 + 14bx^2$, $3x^2 - (3m + n - 3)x - 3m - n$, $x^4 - 2x^2 + 1$, and $2x^2 + (2p + q + 2)x + 2p + q$.

9. $x^4 - 27b^3x$, $(x^2 - 3bx)^2$, $a^3x - a^2bx - 6ab^2x$, and $a^2bx^2 - 4ab^2x^2 + 3b^3x^2$.

10. $4x^{4n} - 2x^{3n} + 3x^n - 9$ and $2x^{4n} + x^{2n} - 2x^{3n} + 3x^n - 6$.

11. $(a + b)(a - b)$, $(a + b)(b - a)$, and $(b + a)^2(a - b)^2$.

12. $2b^3 - 10ab^2 + 8a^2b$, $4a^2 - 5ab + b^2$, $a^4 - b^4$, $9a^4 - 3ab^3 + 3a^2b^2 - 9a^3b$, and $3a^3 - 3a^2b + ab^2 - b^3$.

13. $3x^{3n} - 3mx^{2n} + 2m^2x^n - 2m^3$ and $3x^{3n} + 2m^2x^n + 8m^3 + 12mx^{2n}$.

14. $(m - n)(x - y)$, $(m - n)(y - x)$, and $(n - m)(x - y)$.

15. $9x^2 + 3x^3 + 12x + 20 + x^4$, $3x^2y^2 + x^2y + 2x^2 + 12y^2 + 4y + 8$, $6x^2 + x^5 + 6x^3 + 8x + 24$, and $x^2y^2 + 3x^2y + 4x^2 + 4y^2 + 12y + 16$.

16. $a^{3n} + 3a^{2n}b^m + 3a^nb^{2m} + b^{3m}$, $5a^{5n} + 5b^{5m}$, $4a^{2n}b^{2m} + 12a^nb^{3m} + 8b^{4m}$, and $a^{2n} - b^{2m}$.

17. $x^4 - mx^3 + (n - 1)x^2 + mx - n$ and $x^4 - nx^3 + (m - 1)x^2 - nx - m$.

18. $3n^2x^2 + 12m^2n^2 + 3nx^3 - 15mn^2x + 12m^2nx - 15mnx^2$ and $mnx^3 + 6m^3n^2 - 2n^2x^3 + 6m^3nx + 2mn^2x^2 - 6m^2n^2x - 2nx^4 - 6m^2nx^2$.

19. $x^4 - m x^3 - m^2 x^2 - m^3 x - 2 m^4$, $x^2 - 6 m^2 + m x$, $x^2 - 2 m^2 - m x$, $3 x^3 - 7 m x^2 + 3 m^2 x - 2 m^3$, and $x^2 - 8 m^2 + 2 m x$.

20. $12x^4 y - 24 x^3 y^2 + 12 x^2 y^3$, $(x^2 y - x y^2)^2$, $x y (x^2 - y^2)^2$, and $8 x^4 y^2 - 24 x^3 y^3 + 24 x^2 y^4 - 8 y^5 x$.

21. $a^3 - 2 a^2 b - a b^2 + 2 b^3$, $a^3 + a^2 b - a b^2 - b^3$, $a^3 - 3 a b^2 + 2 b^3$, $a^6 - b^6$, $3 a c - 3 b c + 2 a b - 2 b^2$, $a^4 - b^4$, and $2 b^2 + a c - b c - 2 a b$.

22. $a^2 - (b + c)^2$, $(a + c)^2 - b^2$, $c^2 - (a + b)^2$, and $a^2 + 2 a b + b^2 + 2 b c + c^2 + 2 a c$.

23. $x^{6n} + x^{3n} - 56$, $x^{4n} + 5 x^{3n} + 6 x^{2n}$, $x^{6n} - 4 x^{3n} - 96$, $x^{3n} + 3 x^{2n} + 3 x^n + 2$, $x^{4n} - 9 x^{2n} + 20$, and $3 x^{3n} + 8 x^{2n} + 5 x^{2n} + 2$.

24. $x^3 - 2 x^2 + 3 x - 6$ and $x^4 - x^3 - x^2 - 2 x$.

25. $4 x y^2 - 2 y^3 + 6 x^2 y$ and $4 x^2 y + 8 x^3 - 4 x y^2$.

26. $35 x^3 + 47 x^2 + 13 x + 1$ and $42 x^4 + 41 x^3 - 9 x^2 - 9 x - 1$.

27. $m n^3 + 2 m n^2 + m n + 2 m$ and $3 n^5 - 12 n^3 - 3 n^2 - 6 n$.

28. $2 m^2 y^5 + 166 m^2 y^2 - 96 m^2 y + 108 m^2$ and $6 m n^2 y^5 - 144 m n^2 y^3 - 78 m n^2 y^2 - 108 m n^2$.

29. $2 x^4 - 6 x^3 + 3 x^2 - 3 x + 1$ and $x^7 - 3 x^6 + x^5 - 4 x^2 + 12 x - 4$.

30. $4 x^6 + 32 x^3 + 36 x^2 + 8 x$ and $8 x^6 - 24 x^4 + 24 x^2 - 8$.

31. $x^{3n} - 8 y^{3m} x^{2n} - x^n y^{2m} + 2 y^{2m}$ and $x^{2n} - 4 x^n y^m + 4 y^{2m}$.

CHAPTER XIII.

LOWEST COMMON MULTIPLE.

67. A **Multiple** of a number contains all the factors of the given number with *highest* powers.

Thus, since $24 = 2^3 \times 3$, $2^3 \times 3$ is a multiple of 24.

A **Common Multiple** of two or more numbers contains all the factors of the given numbers with *highest* powers.

Thus, since $12 = 2^2 \times 3$ and $9 = 3^2$, $2^2 \times 3^2$ is a common multiple of 12 and 9.

The **Lowest Common Multiple** (L. C. M.) of two or more algebraic expressions is the expression of **lowest degree** which can be exactly divided by each of them.

Thus, $6\,a^4x^3y^2$ is the L. C. M. of $6\,a^4$, $x\,y^2$, x^3, and a^2y^2.

EXAMPLE 1. Find the L. C. M. of $42\,a^5xy^4$, $56\,a\,x^4y^5$, $63\,a^3x^5y^3$, and $21\,a^4x^3y$.

Solution. Separating the expressions into their factors, we have

$$\begin{aligned}
42\,a^5x\,y^4 &= 2\times 3\times 7\times a^5\times x\times y^4,\\
56\,a\,x^4y^5 &= 2^3\times 7\times a\times x^4\times y^5,\\
63\,a^3x^5y^3 &= 3^2\times 7\times a^3\times x^5\times y^3,\\
21\,a^4x^3y &= 3\times 7\times a^4\times x^3\times y.
\end{aligned}$$

$2^3 \times 3^2 \times 7$ is the *least* common multiple of the coefficients 42, 56, 63, and 21 ; a^5 is the *lowest* power of a that can be evenly divided by each of the factors a^5, a, a^3, a^4 ; x^5 is the *lowest* power of x that can be evenly divided by each of the factors x, x^4, x^5, x^3 ; y^5 is the *lowest* power of y that can be evenly divided by each of the factors y^4, y^5, y^3, y. Hence, the L. C. M. $= 2^3 \times 3^2 \times 7 \times a^5 \times x^5 \times y^5 = 504\,a^5x^5y^5$.

EXAMPLE 2. Find the L. C. M. of $6x^2-2x$, $9x^2-3x$, $6(x^2+xy)$, $8(xy+y^2)^2$, and $12a^2x^3y^2$.

Solution. Separating the given expressions into their factors, we have

$$\begin{array}{rllllll}
12a^2x^3y^2 = & 2^2 & \times 3 & \times a^2 & \times x^3 & \times y^2, & \\
8(xy+y^2)^2 = & 2^3 & & & & \times y^2 \times (x+y)^2, & \\
6(x^2+xy) = & 2 & \times 3 & & \times x & \times (x+y), & \\
6x^2-2x = & 2 & & & \times x & & \times (3x-1), \\
9x^2-3x = & & 3 & & \times x & & \times (3x-1).
\end{array}$$

$2^3 \times 3$ is the *least* common multiple of the coefficients; a^2 is the *lowest* power of a that can be evenly divided by a^2; x^3 is the *lowest* power of x that can be evenly divided by each of the factors x^3, x, x, x. Similarly y, $(x+y)$, and $(3x-1)$, each affected with its *highest* exponent, must be used as multipliers.

$$\begin{aligned}\text{Therefore, the L. C. M.} &= 2^3 \times 3 \times a^2 \times x^3 \times y^2 \times (x+y)^2 \times (3x-1)\\ &= 24a^2x^3y^2(x+y)^2(3x-1).\end{aligned}$$

EXAMPLE 3. Find the L.C.M. of $4ax^2y^2+11axy^2-3ay^2$, x^3+6x^2+9x, $3x^3y^3+7x^2y^3-6xy^3$, and $24ax^2-22ax+4a$.

Process.

$$\begin{array}{rllll}
4ax^2y^2+11axy^2-3ay^2 = & a & \times y^2(x+3) & \times (4x-1), & \\
x^3+6x^2+9x = & x \times & (x+3)^2, & & \\
3x^3y^3+7x^2y^3-6xy^3 = & x & y^3(x+3) \times & & (3x-2), \\
24ax^2-22ax+4a = & 2a & & \times (4x-1) & (3x-2).
\end{array}$$

$\therefore$ the L. C. M. $= 2axy^3(x+3)^2(4x-1)(3x-2)$. Hence, in general,

To Find the L. C. M. of Two or more Expressions that can be Factored by Inspection. Separate the expressions into their factors. Take the product of the factors affecting each with its highest exponent.

Note. The L. C. M. of two or more prime expressions is their product. Thus, the L. C. M. of

a^2+ab+b^2, a^3+b^3, and a^2+b^2 is $(a^2+ab+b^2)(a^2+b^2)(a^3+b^3)$.

Exercise 60.

Find the L. C. M. of:

1. $48x^3y^2$, $56ax^4y^3$, and $63y^2z^3$.

2. $24mn^2x^4$, $36m^3n^2x^2$, and $48n^3z^3$.

3. $18a^2b^2c^3$, $9a^3bc^2$, and $36ab^3c^4$.

4. $12m^4n^2y^3$, $18mny^5$, and $24m^5n^3$.

5. $12ax^3y^4$, x^2-y^2, $x^2-2xy+y^2$, and $x^2+2xy+y^2$.

6. $m^2(x^2-y^2)$, $n^2(x-y)$, and x^4-y^4.

7. $2x(x-y)$, $4xy(x^2-y^2)$, and $6xy^2(x+y)$.

8. x^2+x-20, $x^2-10x+24$, and x^2-x-30.

9. x^2+2x, x^2+4x+4, x^2+3x+2, and x^2+5x+6.

10. $x^4+a^2x^2+a^4$ and $x^4-ax^3-a^3x+a^4$.

11. $x^2-3x-28$, x^2+x-12, and $x^2-10x+21$.

12. $15(x^2y-xy^2)$, $21(x^3-xy^2)$, and $35(xy^2+y^3)$.

13. x^2-1, x^3+1, and x^3-1.

14. $3x^2+11x+6$, $3x^2+8x+4$, and x^2+5x+6.

15. $x^2+(a+b)x+ab$, $x^2+(a+c)x+ac$, and $x^2+(b+c)x+bc$.

16. $mx-my-nx+ny$, $(x-y)^2$, and $3m^2n-3mn^2$.

17. $x^2+(a+b)x+ab$ and $x^2+(a-b)x-ab$.

18. x^2-1, x^2+1, x^4+1, and x^8+1.

19. $x^3+x^2y+xy^2+y^3$, $x^3-x^2y+xy^2-y^3$, and $x^3+x^2y-xy^2-y^3$.

20. $6\,a\,x^3 + 7\,a^2\,x^2 - 3\,a^3\,x$, $3\,a^2\,x^2 + 14\,a^3\,x - 5\,a^4$, and $6\,x^2 + 39\,a\,x + 45\,a^2$.

21. $x^2 + 5\,x + 4$, $x^2 + 2\,x - 8$, and $x^2 + 7\,x + 12$.

22. $12\,x^2 - 23\,x\,y + 10\,y^2$, $4x^2 - 9x\,y + 5\,y^2$, and $3\,x^2 - 5\,x\,y + 2\,y^2$.

23. $a^2 - 4\,b^2$, $a^3 - 2\,a^2\,b + 4\,a\,b^2 - 8\,b^3$, and $a^3 + 2\,a^2\,b + 4\,a\,b^2 + 8\,b^3$.

24. $a\,m + a\,n + b\,m + b\,n$ and $a\,x + a\,y + b\,x + b\,y$.

25. $8\,x^2 - 38\,xy + 35\,y^2$, $4\,x^2 - x\,y - 5\,y^2$, and $2\,x^2 - 5\,x\,y - 7\,y^2$.

26. $x^2 + y^2$, $x^4 - x^2\,y^2 + y^4$, and $x^6 + y^6$.

27. $60\,x^4 + 5\,x^3 - 5\,x^2$, $60\,x^2\,y + 32\,x\,y + 4\,y$, and $40\,x^3\,y - 2\,x^2\,y - 2\,x\,y$.

28. $(a + b)^2 - (c + d)^2$, $(a + c)^2 - (b + d)^2$, and $(a + d)^2 - (b + c)^2$.

29. $x^2 + x\,y + y^2$, $x^2 - x\,y + y^2$, and $x^4 + x^2\,y^2 + y^4$.

30. $3\,x^4 + 26\,x^3 + 35\,x^2$, $6\,x^2 + 38\,x - 28$, and $27\,x^3 + 27\,x^2 - 30\,x$.

31. $12\,x^{2n} + 3\,x^n - 42$, $12\,x^{3n} + 30\,x^{2n} + 12\,x^n$, and $32\,x^{2n} - 40\,x^n - 28$.

32. $a\,(m - n)$, $b\,(n - m)$, and $-\,c\,(m - n)$.

33. $(a - b)\,(b - c)$, $(b - a)\,(b - c)$, and $(b - a)(c - b)$.

34. $a\,(b - x)(x - c)$, $b\,(c - x)\,(x - a)$, and $c\,(a - x)\,(x - b)$.

35. $x^{4n} - 2\,x^{2n} + 1$ and $x^{4n} + 4\,x^{3n} + 6\,x^{2n} + 4\,x^n + 1$.
Result. $x^{6n} + 2\,x^{5n} - x^{4n} - 4\,x^{3n} - x^{2n} + 2\,x^n + 1$.

68. If the expressions cannot be factored by inspection, find their H. C. F., then proceed as before. Thus,

EXAMPLE 1. Find the L. C. M. of $2x^4 + x^3 - 20x^2 - 7x + 24$ and $2x^4 + 3x^3 - 13x^2 - 7x + 15$.

Solution. The H. C. F. of the expressions (Art. 64) is $x^2 + 2x - 3$. Dividing each expression (for the other factor) by $x^2 + 2x - 3$, we have $2x^2 - 3x - 8$ and $2x^2 - x - 5$. Hence,

$$2x^4 + x^3 - 20x^2 - 7x + 24 = (x^2 + 2x - 3)(2x^2 - 3x - 8),$$
$$2x^4 + 3x^3 - 13x^2 - 7x + 15 = (x^2 + 2x - 3)(2x^2 - x - 5).$$

$$\therefore \text{ the L. C. M.} = (x^2 + 2x - 3)(2x^2 - 3x - 8)(2x^2 - x - 5).$$

EXAMPLE 2. Find the L. C. M. of $x^3 - 8x^2 + 19x + 12$, $x^3 - 6x^2 + 11x - 6$, and $x^3 - 9x^2 + 26x - 24$.

Solution. The H. C. F. of the expressions (Art. 66) is $x - 3$. Dividing each of the expressions by $x - 3$, and factoring the quotients, we have

$$x^3 - 8x^2 + 19x - 12 = (x-3)(x^2 - 5x + 4) = (x-3)(x-1)(x-4),$$
$$x^3 - 6x^2 + 11x - 6 = (x-3)(x^2 - 3x + 2) = (x-3)(x-1)(x-2),$$
$$x^3 - 9x^2 + 26x - 24 = (x-3)(x^2 - 6x + 8) = (x-3)(x-2)(x-4).$$

Therefore, the L. C. M. $= (x-3)(x-1)(x-2)(x-4)$
$= x^4 - 10x^3 + 35x^2 - 50x + 24$. Hence,

To Find the L. C. M. of Two or more Polynomials that cannot readily be Factored by Inspection. Find the H. C. F. of the given polynomials, and divide each polynomial by it. Then find the L. C. M. of their quotients, and multiply it by the H. C. F.

Exercise 61.

Find the L. C. M. of:

1. $x^3 + x^2 - 8x - 6$ and $2x^3 - 5x^2 - 2x + 2$.
2. $x^3 + 3x^2 - x - 3$ and $x^3 + 4x^2 + x + 6$.
3. $x^2 + 2x - 3$, $x^3 + 3x^2 - x - 3$, and $x^3 + 4x^2 + x - 6$.

4. $x^4 - m\,x^3 - m^2\,x^2 - m^3\,x - 2\,m^2$ and $3\,x^3 - 7\,m\,x^2 + 3\,m^2\,x - 2\,m^3$.

5. $15\,x^5 + 10\,x^4\,y + 4\,x^3\,y^2 + 6\,x^2\,y^3 - 3\,x\,y^4$ and $12\,x^3\,y^2 + 38\,x^2\,y^3 + 16\,x\,y^4 - 10\,y^5$.

6. $x^3 - 9\,x^2 + 26\,x - 24$, $x^3 - 10\,x^2 + 31\,x - 30$, and $x^3 - 11\,x^2 + 38\,x - 40$.

7. $x^4 - x^3 - 4\,x^2 + 16\,x - 24$, $x^3 - 5\,x^2 + 8\,x - 4$, and $x^2 - 2\,x - 8$.

8. $x^3 + x^2 - 10\,x + 8$, $x^2 + 2\,x - 8$, $x^2 - 3\,x + 2$, and $x^2 - 1$.

9. $6\,x^3 + 15\,x^2 - 6\,x + 9$ and $9\,x^3 + 6\,x^2 - 51\,x + 36$.

10. $2\,x^5 - 8\,x^4 + 12\,x^3 - 8\,x^2 + 2\,x$, $3\,x^5 - 6\,x^3 + 3\,x$, and $x^3 - 3\,x^2 + 3\,x - 1$.

11. $x^4 + 5\,x^3 + 5\,x^2 - 5\,x - 6$, $x^3 + 6\,x^2 + 11\,x + 6$, and $x^3 + 4\,x^2 + x - 6$.

12. $2\,x^3 + 7\,x^2 + 8\,x + 3$, $2\,x^3 - x^2 - 4\,x + 3$, $2\,x^5 + 3\,x^4 + 2\,x^3 + 3\,x^2 + 2\,x + 3$, and $x^4 + x^2 + 1$.

69. To Prove the Method for finding the L. C. M. of any Two or more Algebraic Expressions. Let A, B, D, E, etc. represent the expressions, F represent their H. C. F., and M represent their L. C. M. Also, let a, b, d, e, etc. represent the respective quotients when A, B, D, etc. are divided by F. Then,

$$A = a\,F,\ B = b\,F,\ D = d\,F,\ E = e\,F, \text{ etc.} \tag{1}$$

F is the *product* of *all* the factors *common* to A, B, D, etc. The quotients a, b, d, e, etc. have *no common factor*. Hence, their L. C. M. is $a\,b\,d\,\ldots$, etc. and the L. C. M. of $a\,F$, $b\,F$, $d\,F$, etc., or their equals A, B, D, etc., is $a\,b\,d\,\ldots\,F$. Therefore, $M = a\,b\,d\,e\,F$, etc.

70. Let A, B, D, E, etc. represent any polynomials. Let N represent the L. C. M. of A and B, P the L. C. M. of N and D, and R the L. C. M. of P and E, etc. Evidently R is the expression of *lowest* degree which can be divided by P and E exactly; also, P is the expression of *lowest* degree which is exactly divisible by N and D, and N is the expression of *lowest* degree which is exactly divisible by A and B. Therefore, R is the expression of *lowest* degree which is exactly divisible by A, B, D, and E, etc. Hence,

To Find the L. C. M. of Several Polynomials. Find the L. C. M. of two of them; then of this result and one of the remaining expressions; and so on.

71. Let A and B represent any two expressions. Let F represent their H. C. F., and M represent their L. C. M. Also, let a and b be the respective quotients when A and B are divided by F. Then $A = aF$, $B = bF$, and $M = abF$. Multiplying the first equations together (Axiom 3, Art. 47), we have $A \times B = aF \times bF = F \times abF$. Therefore, substituting for abF its value M, $AB = FM$. Hence, in general,

The Product of any Two Expressions is Equal to the Product of their H. C. F. and L. C. M.

Miscellaneous Exercise 62.

Find the L. C. M. of:

1. $a^3 - 2a^2b + 2ab^2 - b^3$, $a^3 + 2a^2b - ab^2 - 2b^3$, $a^3 + a^2b - ab^2 - b^3$, and $a^3 - 2a^2b - ab^2 + 2b^3$.

2. $x^{4n} - 10x^{2n} + 9$, $x^{4n} + 10x^{3n} + 20x^{2n} - 10x^n - 21$, and $x^{4n} + 4x^{3n} - 22x^{2n} - 4x^n + 21$.

3. $x^{3n} - 4x^{2n}y^m + 9x^ny^{2m} - 10y^{3m}$ and $x^{3n} + 2x^{2n}y^m - 3x^ny^{2m} + 20y^{3m}$.

4. $x^5 + 3x^4 + x^3 + 3x^2 + x + 3$, $2x^3 + 6x^2 - 2x - 6$, $x^5 + 2x^4 + x^3 + 2x^2 + x + 2$, and $2x^5 + 3x^4 + 2x^3 + 3x^2 + 2x + 3$.

5. $xy - bx$, $xy - ay$, $y^2 - 3by + 2b^2$, $xy - 2b^2$, $xy - 2bx - ay + 2ab$, and $xy - bx - ay + ab$.

6. $a^{5n} + a^{4n}b^m + a^{3n}b^{2m} + a^{2n}b^{3m} + a^n b^{4m} + b^{5m}$, and $a^{5n} - a^{4n}b^m + a^{3n}b^{2m} - a^{2n}b^{3m} + a^n b^{4m} - b^{5m}$.

7. $x^{2n} - 4a^{2m}$, $x^{3n} + 2a^m x^{2n} + 4a^{2m}x^n + 8a^{3m}$, and $x^{3n} - 2a^m x^{2n} + 4a^{2m}x^n - 8a^{3m}$.

8. $2x^{3n} + (2a - 3b)x^{2n} - (2b^2 + 3ab)x^n + 3b^3$ and $2x^{2n} - (3b - 2c)x^n - 3bc$.

9. $x^3 - 2x^2 + 4x - 8$, $x^3 + 2x^2 - 4x - 8$, $x^3 - 3x^2 - 4x + 12$, and $x^5 - 3x^4 - 20x^3 + 60x^2 + 64x - 192$.

10. $x^{2n} - (a-b)x^n - ab$, $x^{2n} - (b-c)x^n - bc$, $x^{8n} - x^{2n}b^6 - x^{6n}b^2 + b^8$, and $x^{2n} - (c-a)x^n - ac$.

Find the H.C.F. and L.C.M. of:

11. $3x^3 - 7x^2y + 5xy^2 - y^3$, $x^2y + 3xy^2 - 3x^3 - y^3$, and $3x^3 + 5x^2y + xy^2 - y^3$.

12. $6x^5 + 15x^4y - 4a^3x^3 - 10a^2x^2y$ and $9x^3y - 27ax^2y - 6a^2xy + 18a^3y$.

13. $6x^{3n} + x^{2n} - 5x^n - 2$ and $6x^{3n} + 5x^{2n} - 3x^n - 2$.

14. $a^2 - ab + b^2$, $a^2 + ab + b^2$, $a^4 + a^2b^2 + b^4$, $a^3 + b^3$, $a^3 - b^3$, and $(a^2 - b^2)^2$.

15. $2x^2 + (6a - 10b)x - 30ab$ and $3x^2 - (9a + 15b)x + 45ab$.

16. $x^{3n} - 9x^{2n} + 26x^n - 24$ and $x^{3n} - 12x^{2n} + 47x^n - 6$.

17. $(x^2 + b^2)c + (b^2 + c^2)x$ and $(x^2 - b^2)c + (b^2 - c^2)x$.

18. $(2x^2 - 3m^2)y + (2m^2 - 3y^2)x$ and $(2m^2 + 3y^2)x + (2x^2 + 3m^2)y$.

19. $x^{3n} + 2a^m x^{2n} + a^{2m} x^n + 2a^{3m}$ and $x^{3n} - 2a^m x^{2n} + a^{2m} x^n - 2a^{3m}$.

20. $x^3 - 3x^2y + 3xy^2 - 2y^3$, $x^3 - x^2y - xy^2 - 2y^3$, $x^2 + xy + y^2$, and $x^4 + x^2y^2 + y^4$.

21. $20x^4 + x^2 - 1$, $25x^4 + 5x^3 - x - 1$, and $25x^4 - 10x^2 + 1$.

22. $x^{3n} - y^{3m}$, $x^{3n}y^m - y^{4m}$, $y^{2m}(x^n - y^m)^2$, and $x^{2n} + x^n y^m + y^{2m}$.

Find the L.C.M. of:

23. $x^4 - 7x^3 - 7x^2 + 43x + 42$ and $x^4 - 9x^3 + 9x^2 + 41x - 42$.

24. $x^3 + 4x^2 + 6x + 9$, $x^3 + x^2 - 2x + 12$, and $x^2 - x - 12$.

25. $4x^6 - 4x^4 - 29x^2 - 21$ and $4x^6 + 24x^4 + 41x^2 + 21$.

26. $2x^4 - 11x^3 + 3x^2 + 10x$ and $3x^4 - 14x^3 - 6x^2 + 5x$.

27. $x^3 - 6x^2 + 11x - 6$, $x^3 - x^2 - 14x + 24$, and $x^3 + x^2 - 17x + 15$.

28. $3x^4 + 5x^3 + 5x^2 + 5x + 2$ and $3x^4 - x^3 + x^2 - x - 2$.

29. $9x^4 + 18x^3 - x^2 - 9x + 4$ and $6x^4 + 17x^3 - 10x + 8$.

30. $2m^3 + m^2 - m + 3$ and $2m^3 + 5m^2 - m - 6$.

CHAPTER XIV.

ALGEBRAIC FRACTIONS.

72. THE expression $(a + b) \div (m + n)$ may be written $\frac{a+b}{m+n}$. It is read the same in each case.

The second form is called a **Fraction**; the dividend is the **Numerator**, the divisor is the **Denominator**, and the two taken together are called the **Terms** of the fraction. a, b, m, and n may represent any numbers whatever. Hence,

An **Algebraic Fraction** is an indicated operation in division.

A **Mixed Expression** is one composed of entire and fractional parts; as,

$$m + \frac{n}{a}.$$

Note. The dividing line has the force of a symbol of aggregation, and the sign before it is the sign of the fraction and belongs to its algebraic value.

73. Multiplying or dividing the divisor and the dividend by the same number does not change the quotient. For, if we multiply the dividend by any number, as m, the quotient will be increased m times; if we multiply the divisor by m, the quotient will be diminished as many times. A similar method of reasoning may be applied to the dividend and divisor.

A fraction is in its **lowest terms** when the numerator and denominator have no *common factor*.

EXAMPLE 1. Reduce $\frac{7a^2bc}{28a^3bc^2}$ to its lowest terms.

Solution. The H.C.F. of the numerator and the denominator is $7a^2bc$. Factoring, we have $\frac{7a^2bc \times 1}{7a^2bc \times 4ac}$. Rejecting the H.C.F., we have $\frac{1}{4ac}$. Since the terms are prime to each other the fraction is in its lowest form.

EXAMPLE 2. Reduce $\frac{6a^2 + ax - 15x^2}{15a^2 + 16ax - 15x^2}$ to its lowest terms.

Process.
$$\frac{6a^2 + ax - 15x^2}{15a^2 + 16ax - 15x^2} = \frac{(3a+5x)(2a-3x)}{(3a+5x)(5a-3x)}$$
$$= \frac{2a-3x}{5a-3x}.$$

Explanation. Dividing the terms of the fraction by their H.C.F., we have $\frac{2a-3x}{5a-3x}$. This result is in its lowest terms, since the numerator and denominator have no common factor.

EXAMPLE 3. Reduce $\frac{x^4 + x^3y + xy^3 - y^4}{x^4 - x^3y - xy^3 - y^4}$ to its lowest terms.

Process.
$$\frac{x^4+x^3y+xy^3-y^4}{x^4-x^3y-xy^3-y^4} = \frac{(x^4-y^4)+(x^3y+xy^3)}{(x^4-y^4)-(x^3y+xy^3)}$$
$$= \frac{(x^2+y^2)(x^2-y^2)+xy(x^2+y^2)}{(x^2+y^2)(x^2-y^2)-xy(x^2+y^2)}$$
$$= \frac{(x^2+y^2)[x^2-y^2+xy]}{(x^2+y^2)[x^2-y^2-xy]}$$
$$= \frac{x^2+xy-y^2}{x^2-xy-y^2}.$$

When the factors of the numerator and denominator cannot be readily found by inspection, their H.C.F. may be found by the method of Art. 64, and the fraction then reduced to its lowest terms. Thus,

EXAMPLE 4. Reduce $\frac{4a^3 + 12a^2b - ab^2 - 15b^3}{6a^3 + 13a^2b - 4ab^2 - 15b^3}$ to its lowest terms.

Solution. $6a^3+13a^2b-4ab^2-15b^3)\,4a^3+12a^2b-ab^2-15b^3(\,2$

$$\begin{array}{ll}
\text{3 times the numerator,} & 12a^3+36a^2b-3ab^2-45b^3 \\
\text{2 times the denominator,} & \underline{12a^3+26a^2b-8ab^2-30b^3} \\
\text{First remainder,} & 10a^2b+5ab^2-15b^3 \\
 & =5b(2a^2+ab-3b^2).
\end{array}$$

$$\begin{array}{r}
2a^2+ab-3b^2)\,6a^3+13a^2b-4ab^2-15b^3(\,3a+5b \\
\underline{6a^3+3a^2b-9ab^2} \\
10a^2b+5ab^2-15b^3 \\
\underline{10a^2b+5ab^2-15b^3}
\end{array}$$

$\therefore$ the H. C. F. of the numerator and denominator is $2a^2+ab-3b^2$. Dividing each term of the fraction by $2a^2+ab-3b^2$, we have

$$\frac{4a^3+12a^2b-ab^2-15b^3}{6a^3+13a^2b-4ab^2-15b^3}=\frac{2a+5b}{3a+5b}.$$ Hence,

To Reduce a Fraction to its Lowest Terms. Divide both terms by their H. C. F.

Exercise 63.

Reduce to lowest terms:

1. $\dfrac{15axy^2}{75ax^2y^3}$; $\dfrac{6mx^2y^3}{9m^3x^5y^7}$; $\dfrac{4x^2-9y^2}{4x^2+6xy}$.

2. $\dfrac{72m^5n^{\frac{3}{2}}x^{2n}}{24mn^{\frac{1}{2}}x^n}$; $\dfrac{mn^2(x^3-y^3)^2}{m^2n(x^3-y^3)}$; $\dfrac{2x^2+3x+1}{x^2-x-2}$.

3. $\dfrac{6m^2-11m-10}{6m^2-19m+10}$; $\dfrac{20(x^3-y^3)}{5x^2+5xy+5y^2}$; $\dfrac{x^my^{2n}}{x^{2m}y^{n+1}}$.

4. $\dfrac{3m^2+23m-36}{4m^2+33m-27}$; $\dfrac{3m^4+9m^3n+6m^2n^2}{m^4+m^3n-2m^2n^2}$.

5. $\dfrac{3m+3mx}{4m^{\frac{3}{2}}-4m^{\frac{3}{2}}x^2}$; $\dfrac{x^2-(a+b)x+ab}{x^2+(c-a)x-ac}$.

6. $\dfrac{(m+n)^2-x^2}{mx+nx-x^2}$; $\dfrac{x^3-3x^2y+3xy^2-y^3}{x^3-x^2y-xy^2+y^3}$.

7. $\dfrac{x^2-(y+m)^2}{x^2+xy+mx}$; $\dfrac{ac-ad-bc+bd}{a^3-b^3}$.

8. $\dfrac{x^2+(a+b)x+ab}{x^2+(a+c)x+ac}$; $\dfrac{27a+a^4}{18a-6a^2+2a^3}$.

9. $\dfrac{a^2-(b+c)^2}{(a-b)^2-c^2}$; $\dfrac{(a+x)^2-(b+c)^2}{(b+x)^2-(a+c)^2}$.

10. $\dfrac{m^5-m^4n-mn^4+n^5}{m^4-m^3n-m^2n^2+mn^3}$; $\dfrac{ax^m-bx^{m+1}}{a^2bx-b^3x^3}$.

11. $\dfrac{x^3-4x^2+9x-10}{x^3+2x^2-3x+20}$; $\dfrac{15x^3+35x^2+3x+7}{27x^4+63x^3-12x^2-28x}$.

12. $\dfrac{a^3-a^2b-ab^2-2b^3}{a^3+3a^2b+3ab^2+2b^3}$; $\dfrac{24x^4-22x^2+5}{48x^4+16x^2-15}$.

74. Example. Reduce $\dfrac{4x^3-16x^2y+29xy^2-22y^3}{2x-3y}$ to a mixed expression.

Process.

$$
\begin{array}{l}
2x-3y\,)\,4x^3-16x^2y+29xy^2-22y^3\,(\,2x^2-5xy+7y^2+\dfrac{-y^3}{2x-3y} \\
\qquad\quad \underline{4x^3-\;\;6x^2y} \\
\qquad\qquad -10x^2y+29xy^2 \\
\qquad\qquad \underline{-10x^2y+15xy^2} \\
\qquad\qquad\qquad\quad 14xy^2-22y^3 \\
\qquad\qquad\qquad\quad \underline{14xy^2-21y^3} \\
\qquad\qquad\qquad\qquad\quad -\;\;y^3\,.
\end{array}
$$

Explanation. Dividing the numerator by the denominator, we have $2x^2-5xy+7y^2$ for a quotient, and a remainder of $-y^3$. As $-y^3$ is not exactly divisible by $2x-3y$, we indicate the division and add the result to $2x^2-5xy+7y^2$. Hence,

To Reduce a Fraction to an Entire or Mixed Expression. Divide the numerator by the denominator.

Exercise 64.

Reduce to an entire or mixed expression:

1. $\dfrac{1+2x}{1-3x}$; $\dfrac{2x^4-x^3-9x^2+14}{2x^2-x-3}$.

2. $\dfrac{x^3-2x^2}{x^2-x+1}$; $\dfrac{6a^3-13a^2+6a-6}{3a^2-2a+1}$.

3. $\dfrac{x^3+ax^2-3a^2x-3a^3}{x-2a}$; $\dfrac{x^3+2x^2-12x-13}{x^2+x-12}$.

4. $\dfrac{x^4-2x^2y^2+y^4}{x^2+2xy+y^2}$; $\dfrac{x^2+(m+n+1)x+mn+a}{x+n}$.

5. $\dfrac{6x^3-5x^2+7x-5}{2x+1}$; $\dfrac{x^{3m}-x^{2m}y^n+x^my^{2n}-y^{3n}}{x^m-y^n}$.

75. Every expression may be considered as a fraction whose denominator is *unity*. Thus, $a=\dfrac{a}{1}$; $a^3b-c^2=\dfrac{a^3b-c^2}{1}$.

Example. Reduce $x+y-\dfrac{x^2-y^2-5}{x-y}$ to fractional form.

Process.
$$x+y-\frac{x^2-y^2-5}{x-y}=\frac{x+y}{1}-\frac{x^2-y^2-5}{x-y}$$
$$=\frac{(x+y)\times(x-y)}{1\times(x-y)}-\frac{x^2-y^2-5}{x-y}$$
$$=\frac{x^2-y^2-(x^2-y^2-5)}{x-y}=\frac{5}{x-y}.$$

Explanation. Writing the entire part in the form of a fraction whose denominator is 1, and multiplying both terms of it by $x-y$, we have the third expression. Since the sum or difference of the quotients of two or more expressions divided by a common divisor, is the same as the quotient of the sum or difference of the expressions divided by the same divisor, we have the fourth expression. Uniting like terms, we have the result. Hence,

To Reduce a Mixed Expression to the Form of a Fraction. Multiply the entire part by the denominator; to the product annex the numerator; unite like terms and under the result write the denominator.

Notes: 1. In the above example, since the sign before the *dividing line* indicates subtraction, we must subtract the numerator, $x^2 - y^2 - 5$, from $(x+y)(x-y)$.

2. If the sign of the fraction is —, and the numerator is a polynomial, it will be found convenient to enclose it in a symbol of aggregation before annexing it to the product.

Exercise 65.

Reduce to fractional forms:

1. $a - x + \dfrac{x^2}{a+x}$; $\dfrac{m+n}{m-n} + 1$; $a + b - \dfrac{a^2+b^2}{a-b}$.

2. $m^2 - mn + y^2 - \dfrac{y^3}{m+y}$; $\dfrac{x^3+y^3}{x^2+xy+y^2} - (x-y)$.

3. $mn + \dfrac{a - m^2n^2}{mn}$; $m^3 + m^2 + m + 1 - \dfrac{m^4-1}{m-1}$.

4. $x + 1 + \dfrac{(x-1)^2}{x+1}$; $\dfrac{m^3-n^3}{m^3+n^3} + 1$; $m(x+y) + \dfrac{my^2}{x-y}$.

5. $\dfrac{m^9x}{m^2-x^2} - (m^5x^3 + m^7x)$; $x^m - y^n + \dfrac{x^{2m} + y^{2n} - 5}{x^m + y^n}$.

6. $m + n - \dfrac{2n(3m^2+n^2)}{(m+n)^2}$; $(m+n)^2 - \dfrac{m^4+n^4}{(m-n)^2}$.

7. $x^{2m} - x^m y^n + y^{2n} - \dfrac{x^{4m} + y^{4n}}{x^{2m} + x^m y^n + y^{2n}}$.

8. $x^{2m} + 2x^m y^n + y^{2n} - \dfrac{x^{3m} - 3x^{2m}y^n + 3x^m y^{2n} - y^{3n}}{x^m + y^n}$.

76. It may be shown by multiplication (Art. 22) that:

$$(+a)(+b) = ab; \quad (-a)(-b) = ab.$$
$$(+a)(+b)(+c) = abc; \quad (-a)(-b)(-c) = abc.$$
$$(+a)(+b)(+c)(+d) = abcd; \quad (-a)(-b)(-c)(-d) = abcd, \text{ etc.}$$ Hence,

In an indicated product of any number of factors, *all the signs of any even number of factors may be changed without changing the value of the product.* Thus,

$$(x-y)(y-z) = (y-x)(z-y);$$

$(w-x)(x-y)(y-z) = (x-w)(x-y)(z-y)$, changing the signs of the first and third factors.

Note. In order to multiply a product containing several factors by a given expression the student must be careful to multiply only *one* factor of that product by the expression. Thus, in order to multiply both terms of the fraction $\frac{(a+b)(c+d)}{(m+n)(x+y)}$ by a, we must multiply either $a+b$ or $c+d$ and $m+n$ or $x+y$ by a.

77. It is often convenient to change the order and the signs of the terms of the numerator or denominator, or both. Thus,

Change the order and the signs of the terms of the numerator and denominator of the following fractions:

1. $\frac{b-a}{y-x}$.
2. $\frac{m-n}{(c-b)(x-m)}$.

Solutions: 1. Multiplying both terms of the fraction by -1, we have

$$\frac{b-a}{y-x} = \frac{(b-a)\times -1}{(y-x)\times -1} = \frac{a-b}{x-y}.$$

2. Multiplying the factor $x-m$ and the terms of the numerator by -1, we have

$$\frac{m-n}{(c-b)(x-m)} = \frac{(m-n)\times -1}{(c-b)[(x-m)\times -1]} = \frac{n-m}{(c-b)(m-x)}.$$

Multiplying the factor $c-b$ and the numerator of this fraction by -1, and since adding a negative quotient is the same as subtracting a positive quotient, we have

$$\frac{n-m}{(c-b)(m-x)} = +\frac{(n-m)\times -1}{[(c-b)\times -1](m-x)} = -\frac{+(n-m)}{(b-c)(m-x)}.$$

Change to equivalent fractions having the letters arranged alphabetically, and the first letter of each factor in the numerator and the denominator, positive:

3. $\dfrac{x-m}{(b-a)(a-c)(y-x)}$. 4. $\dfrac{(b-a)(c-a)}{(d-a)(c-b)(n-m)}$.

Solutions: 3. Multiplying the numerator and the factor $y-x$ by -1, we have

$$\frac{x-m}{(b-a)(a-c)(y-x)} = \frac{m-x}{(b-a)(a-c)(x-y)}.$$

Multiplying the numerator and the factor $b-a$ of this result by -1, we have

$$\frac{m-x}{(b-a)(a-c)(x-y)} = -\frac{m-x}{(a-b)(a-c)(x-y)}.$$

4. Multiplying the factors $c-a$ and $n-m$, $b-a$ and $c-b$ by -1, respectively, we have

$$\frac{(b-a)(c-a)}{(d-a)(c-b)(n-m)} = \frac{(a-b)(a-c)}{(d-a)(b-c)(m-n)}.$$

Similarly, $\dfrac{(a-b)(a-c)}{(d-a)(b-c)(m-n)} = -\dfrac{(a-b)(a-c)}{(a-d)(b-c)(m-n)}.$

Therefore, $\dfrac{(b-a)(c-a)}{(d-a)(c-b)(n-m)} = -\dfrac{(a-b)(a-c)}{(a-d)(b-c)(m-n)}.$

Exercise 66.

Change each of the following fractions to four equivalent ones with respect to the signs of letters:

1. $\dfrac{m-n}{a-b}$; $-\dfrac{a-b}{m+n-x}$; $\dfrac{m}{a-b+x}$; $\dfrac{m+n-a}{m-n+a}$.

Change the following fractions to equivalent ones having m and n positive in both terms:

2. $\dfrac{m-a}{b-n}$; $\dfrac{a+m-x}{b-m-y}$; $-\dfrac{a+b-n}{a-b+m}$.

3. $\dfrac{x-m}{y-n}$; $-\dfrac{x-m}{(y-m)(z-n)}$; $+\dfrac{(a-m)(b-m)}{(c-m)(x-n)(y-m)}$.

Change the following fractions to equivalent ones having the letters of the terms arranged alphabetically and the first letter of each factor in the denominator positive:

4. $\dfrac{2x-3-y}{(m-a)(2x-b)(b+a)}$; $-\dfrac{3-c+a}{(y-x)(m-n)(a-c)}$.

5. $\dfrac{(x-m)ba}{xymn(c-b)(b-a)(c-a)}$;

$$\frac{(y-x)yx}{cba(b-a)(z-y)(c-a)(y-x)(n-m)}.$$

78. Fractions having a common denominator are **similar.** Thus, $\dfrac{2c}{ab}$, $\dfrac{1}{ab}$, and $\dfrac{n}{ab}$ are similar.

Example 1. Reduce $\dfrac{2x}{5m^2}$, $\dfrac{3}{mn^3}$, and $\dfrac{5n^2}{4x^5}$ to similar fractions having the lowest common denominator.

Solution. Evidently the lowest common denominator is $20m^2n^3x^5$, the L. C. M. of $5m^2$, mn^3, and $4x^5$. Dividing $20m^2n^3x^5$ by the denominator of each fraction, and multiplying both terms of each fraction by the quotient each by each, we have

$$\frac{2x}{5m^2} = \frac{2x \times 4n^3x^5}{5m^2 \times 4n^3x^5} = \frac{8n^3x^6}{20m^2n^3x^5};$$

$$\frac{3}{mn^3} = \frac{3 \times 20mx^5}{mn^3 \times 20mx^5} = \frac{60mx^5}{20m^2n^3x^5};$$

$$\frac{5n^2}{4x^5} = \frac{5n^2 \times 5m^2n^3}{4x^5 \times 5m^2n^3} = \frac{25m^2n^5}{20m^2n^3x^5}.$$

Example 2. Reduce $\frac{x-1}{x^2-8x+15}$, $\frac{x+3}{x^2-4x-5}$, and $\frac{x-5}{x^2+4x+3}$ to similar fractions with lowest common denominator.

Solution. The lowest common denominator is $(x-3)(x-5)(x+1)(x+3)$, the L.C.M. of the denominators. Dividing the L.C.M. by the denominator of each fraction, and multiplying both terms of each fraction by the quotient each by each, we have

$$\frac{x-1}{x^2-8x+15} = \frac{(x-1) \times (x+1)(x+3)}{(x-3)(x-5) \times (x+1)(x+3)} = \frac{(x+3)(x^2-1)}{(x+1)(x-5)(x^2-9)};$$

$$\frac{x+3}{x^2-4x-5} = \frac{(x+3) \times (x-3)(x+3)}{(x-5)(x+1) \times (x-3)(x+3)} = \frac{(x+3)^2(x-3)}{(x+1)(x-5)(x^2-9)};$$

$$\frac{x-5}{x^2+4x+3} = \frac{(x-5) \times (x-5)(x-3)}{(x+3)(x+1) \times (x-5)(x-3)} = \frac{(x-5)^2(x-3)}{(x+1)(x-5)(x^2-9)}.$$

Hence, in general,

To Reduce Fractions to Equivalent Fractions having the Lowest Common Denominator (L. C. D.). Find the L.C.M. of the denominators. Then multiply both terms of each fraction by the quotient of the L.C.M. divided by the denominator of that fraction.

Notes: 1. When the denominators have no common factors, the multiplier for both terms of each fraction will be the product of the denominators of all the other fractions.

2. In all operations with fractions it is better to separate the denominators into their factors at once; and sometimes it is also convenient to factor the numerators.

3. It will be observed that the terms of each fraction are multiplied by an expression which is obtained by dividing the L. C. D. by its own denominator. It is not necessary to state how the multiplier is obtained in every expression.

Exercise 67.

Reduce to similar fractions with L. C. D.:

1. $\frac{a}{b}, \frac{m}{n}, \frac{x}{y}, \frac{abc}{mn}; \quad \frac{1}{ab}, \frac{2}{ac}, \frac{5}{bc}.$

2. $\frac{m+n}{ab}, \frac{m-n}{bc}, \frac{n}{ac}; \quad a, \frac{a}{b}, \frac{a-n}{m}, \frac{n}{3a}.$

3. $\frac{m+2n}{3m}, \frac{2m-3n}{6n}, \frac{5m-n}{10mn}.$

4. $\frac{1}{x^2-1}, \frac{x+2}{x^2+x-2}, \frac{x-2}{x^2-x-2}.$

5. $\frac{m-n}{m+n}, \frac{m+2n}{m-n}, \frac{m^2}{m^2-n^2}; \quad \frac{1}{a+b}, \frac{2}{a-b}, \frac{4}{a^2+b^2}.$

6. $\frac{8m+2}{m-2}, \frac{2m-1}{3m-6}, \frac{3m+2}{5m-10}.$

7. $\frac{xy}{mx-my+nx-ny}, \frac{m-n}{2x^2-2xy}.$

8. $\frac{m}{m+x}, \frac{n}{m^3+x^3}, \frac{a}{m^2-mx+x^2}.$

9. $\frac{x}{x^2-xy+y^2}, \frac{y}{x^2+xy+y^2}, \frac{m}{x^4+x^2y^2+y^4}.$

10. $\frac{x-y}{x^2+xy+y^2}, \frac{x^2+y^2}{x^3-y^3}, \frac{y}{x-y}, \frac{x^3+y^3}{5}.$

11. $\frac{x^m+y^n}{x^{4m}-y^{6n}}, \frac{x^m y^n}{x^{2m}+y^{3n}}, \frac{x^{2m}+y^{2n}}{x^{2m}-y^{3n}}.$

12. $\dfrac{a}{(a+x)^2-b^2}, \; \dfrac{b}{(b+x)^2-a^2}, \; \dfrac{x}{x^2-(a+b)^2}.$

13. $\dfrac{a}{(c-a)(b-c)}, \; \dfrac{b}{(a-c)(c-b)}, \; \dfrac{c}{(c-a)(c-b)}.$

Suggestion. $\dfrac{a}{(c-a)(b-c)} = \dfrac{a \times -1}{[(c-a)\times -1](b-c)} = \dfrac{-a}{(a-c)(b-c)}.$ Etc.

14. $\dfrac{3a}{3-a}, \; \dfrac{4a}{a-3}, \; \dfrac{5a}{(a-3)^2}; \; \dfrac{m}{1-m}, \; \dfrac{n}{m-1}, \; \dfrac{y}{1-m^2}.$

15. $\dfrac{1}{(2-x)(3-x)}, \; \dfrac{2}{(x-1)(2-x)}, \; \dfrac{3}{(x-2)(1-x)}, \; \dfrac{4}{(x-1)(x-2)}.$

Suggestion. $\dfrac{1}{(2-x)(3-x)} = \dfrac{1}{(x-2)(x-3)} =$ etc.

16. $\dfrac{m}{(m-x)(x-n)}, \; \dfrac{x}{(x-m)(a-x)}, \; \dfrac{a}{(x-a)(n-x)}.$

17. $\dfrac{1+x}{(1-x)(2-x)(x-5)}, \; \dfrac{2+x}{(x-1)(2-x)(3-x)(5-x)}.$

18. $\dfrac{x-3}{4-x^2}, \; \dfrac{x-2}{x^2+x-6}, \; \dfrac{x^2+4}{9-6x+x^2}, \; \dfrac{2}{x^2-x-6}.$

19. $\dfrac{x^m}{x^{4m}-1}, \; \dfrac{x^{2m}+1}{x^{4m}+4x^{2m}+3}, \; \dfrac{x^{2m}-1}{x^{4m}+2x^{2m}-3}.$

79. Example 1. Find the sum of $\dfrac{a}{b}$, $\dfrac{c}{d}$, and $\dfrac{m}{n}$.

Solution. Multiplying the terms of the first fraction by dn, of the second by bn, of the third by bd, and adding the results (Arts. 32, 14), we have

$$\frac{a}{b}+\frac{c}{d}+\frac{m}{n}=\frac{adn}{bdn}+\frac{bcn}{bdn}+\frac{bdm}{bdn}=\frac{adn+bcn+bdm}{bdn}.$$

EXAMPLE 2. Subtract $\frac{m}{n}$ from $\frac{a}{b}$.

Solution. Multiplying the terms of the first fraction by n, of the second by b, and subtracting (Art. 19), we have

$$\frac{a}{b} - \frac{m}{n} = \frac{an}{bn} - \frac{bm}{bn} = \frac{an - bm}{bn}.$$

EXAMPLE 3. Subtract $\frac{2a - 3b}{2a}$ from $\frac{3x - 2b}{3x}$.

Solution. Reducing to similar fractions with L. C. D., we have

$$\begin{aligned}\frac{3x - 2b}{3x} - \frac{2a - 3b}{2a} &= \frac{6ax - 4ab}{6ax} - \frac{6ax - 9bx}{6ax}\\ &= \frac{6ax - 4ab - (6ax - 9bx)}{6ax}\\ &= -\frac{b(4a - 9x)}{6ax}.\end{aligned}$$

EXAMPLE 4. Find the sum of $a + \frac{2x - my}{m}$ and $b - \frac{3x - ny}{n}$.

Solution. Uniting the entire parts, and reducing to similar fractions, we have

$$\begin{aligned}\left(a + \frac{2x - my}{m}\right) + \left(b - \frac{3x - ny}{n}\right) &= a + b + \frac{(2x - my)n}{mn} - \frac{(3x - ny)m}{mn}\\ &= a + b + \frac{(2x - my)n - (3x - ny)m}{mn}\\ &= a + b + \frac{(2n - 3m)x}{mn}.\end{aligned}$$

Note 1. If the sign of a fraction is $-$, care must be taken to change the sign of each term in the numerator before combining it with the others. In such case the beginner should enclose the numerator in parentheses, as shown in the above work.

EXAMPLE 5. Simplify $\frac{2x - 6}{x^2 + 3x + 2} - \frac{x + 2}{x^2 - 2x - 3} - \frac{x + 1}{x^2 - x - 6}$.

Process. $\frac{2x-6}{x^2+3x+2}-\frac{x+2}{x^2-2x-3}-\frac{x+1}{x^2-x-6}$

$$=\frac{2(x-3)}{(x+1)(x+2)}-\frac{x+2}{(x+1)(x-3)}-\frac{x+1}{(x+2)(x-3)}$$

$$=\frac{2(x-3)\times(x-3)}{(x+1)(x+2)\times(x-3)}-\frac{(x+2)\times(x+2)}{(x+1)(x-3)\times(x+2)}-\frac{(x+1)\times(x+1}{(x+2)(x-3)\times(x+1)}$$

$$=\frac{2(x-3)^2-(x+2)^2-(x+1)^2}{(x+1)(x+2)(x-3)}=\frac{13-18x}{(x+1)(x+2)(x-3)}.$$

Notes: 2. In finding the value of an expression like $-(x+2)^2$, the beginner should first express the product in a parentheses and then remove the parentheses as above.

3. Sometimes it is better not to reduce all the fractions to the L. C. D. at once. Thus,

Example 6. $\frac{1}{x-2y}-\frac{4}{x-y}+\frac{6}{x}-\frac{4}{x+y}+\frac{1}{x+2y}$

$$=\frac{1}{x-2y}+\frac{1}{x+2y}-\frac{4}{x-y}-\frac{4}{x+y}+\frac{6}{x}$$

$$=\frac{x+2y}{(x-2y)(x+2y)}+\frac{x-2y}{(x+2y)(x-2y)}-\frac{4(x+y)}{(x-y)(x+y)}-\frac{4(x-y)}{(x+y)(x-y)}+\frac{6}{x}$$

$$=\frac{2x}{x^2-4y^2}-\frac{8x}{x^2-y^2}+\frac{6}{x}$$

$$=\frac{2x(x^2-y^2)}{(x^2-4y^2)(x^2-y^2)}-\frac{8x(x^2-4y^2)}{(x^2-y^2)(x^2-4y^2)}+\frac{6}{x}$$

$$=\frac{30xy^2-6x^3}{(x^2-4y^2)(x^2-y^2)}+\frac{6}{x}$$

$$=\frac{(30xy^2-6x^3)x}{(x^2-4y^2)(x^2-y^2)x}+\frac{6(x^2-4y^2)(x^2-y^2)}{x(x^2-4y^2)(x^2-y^2)}$$

$$=\frac{24y^4}{x(x^2-4y^2)(x^2-y^2)}.$$ Hence, in general,

To Add or Subtract Fractions. Reduce to similar fractions with L. C. D.; add or subtract the numerators, and divide the result by their L. C. D.

Exercise 68.

Simplify:

1. $\dfrac{2a-5}{12a}+\dfrac{3a-11}{18}$; $\dfrac{b+c}{4a}+\dfrac{a+c}{8b}-\dfrac{a-b}{9c}$.

2. $\dfrac{2x+5}{x}-\dfrac{x+3}{2x}-\dfrac{27}{8x^3}$; $\dfrac{2}{xy}-\dfrac{3y^2-x^2}{xy^3}+\dfrac{xy+y^2}{x^2y^2}$.

3. $\dfrac{p}{ab}+\dfrac{m}{ac}-\dfrac{n}{bc}$; $\left(\dfrac{m+n}{n}+\dfrac{a+b}{a}\right)-\left(\dfrac{m-n}{3n}+\dfrac{a-b}{4a}\right)$.

4. $\dfrac{3+m}{n}+\dfrac{4-am}{an}+\dfrac{a}{3n}$; $\left(\dfrac{5}{m}-\dfrac{4}{n}+\dfrac{3}{x}\right)+\left(\dfrac{1}{m}-\dfrac{2}{n}-\dfrac{3}{x}\right)$.

5. $\dfrac{5a-b}{2b}+\dfrac{7a+3b}{6b}-\left(\dfrac{2a}{b}+\dfrac{a-b}{3b}\right)$.

6. $\left(m+\dfrac{m}{n}\right)+\left(3m-\dfrac{a}{n}\right)-\left(4m+\dfrac{b}{n}\right)$.

7. $\dfrac{a^2-bc}{bc}-\dfrac{ac-b^2}{ac}-\dfrac{ab-c^2}{ab}$; $\dfrac{2a^2-b^2}{a^2}-\dfrac{b^2-c^2}{b^2}-\dfrac{c^2-a^2}{c^2}$.

8. $\left(m+n-\dfrac{n}{mx}\right)-\left(2m-3n+\dfrac{m}{nx}\right)$.

9. $\left(\dfrac{3}{5x}-\dfrac{4}{3y}+\dfrac{5}{6m}\right)-\left(\dfrac{8}{3x}-\dfrac{7}{10y}-\dfrac{2}{7m}\right)$.

10. $\dfrac{a+b}{c}+\dfrac{b-c}{a}-\dfrac{c-a}{b}-\dfrac{ab^2-bc^2-ca^2}{abc}$.

11. $\dfrac{1}{x-5}-\dfrac{1}{x-4}$; $\dfrac{x+2}{x-2}-\dfrac{x-2}{x+2}$; $\dfrac{3}{2m(m-1)}-\dfrac{5}{4m(m-2)}$.

12. $\dfrac{2\,am-3\,bn}{3\,mn\,(m-n)}-\dfrac{2\,am+3\,bn}{3\,mn\,(m+n)};\quad \dfrac{1}{x^2-4\,x+4}-\dfrac{1}{x^2+x-6}.$

13. $\dfrac{x^2+xy+y^2}{x+y}+\dfrac{x^2-xy+y^2}{x-y};\quad \dfrac{1}{x-m}+\dfrac{1}{x+m}-\dfrac{2}{x}.$

14. $\dfrac{1}{m+3}+\dfrac{m+3}{m-4}-\dfrac{-7}{m^2-m-12};\quad \dfrac{2}{m}-\dfrac{3}{2\,m-1}-\dfrac{2\,m-3}{4\,m^2-1}.$

15. $\dfrac{m}{m+n}+\dfrac{n}{m-n}+\dfrac{2\,mn}{m^2-n^2};\quad \dfrac{1}{a-2\,x}-\dfrac{(a+2\,x)^2}{a^3-8\,x^3}.$

16. $\dfrac{x}{xy-y^2}-\dfrac{1}{x-y}-\dfrac{1}{y};\quad \dfrac{1}{m^2-(n+x)^2}+\dfrac{1}{x^2-(m+n)^2}.$

17. $\dfrac{x^4+3\,x^2y^2+y^4}{x^3-y^3}\quad \dfrac{x^2-xy+y^2}{x-y};\quad \dfrac{2\,x^2}{x^4-y^4}-\dfrac{x+y}{x^3+x^2y+xy^2+y^3}.$

18. $\dfrac{x+4}{x^2+5\,x+6}+\dfrac{x+3}{x^2+6\,x+8}-\dfrac{x+2}{x^2+7\,x+12}.$

19. $\dfrac{1}{m+n}+\dfrac{mn}{m^3+n^3}+\dfrac{m-n}{m^2-mn+n^2}.$

20. $\dfrac{1}{m+x}+\dfrac{1}{m-x}-\dfrac{x}{(m+x)^2}+\dfrac{x}{(m-x)^2}.$

21. $\dfrac{1}{8-8\,x}-\dfrac{1}{8+8\,x}+\dfrac{x}{4+4\,x^2}+\dfrac{x}{2+2\,x^4}.$

22. $\dfrac{24\,x}{9-12\,x+4\,x^2}-\dfrac{3+2\,x}{3-2\,x}+\dfrac{3-2\,x}{3+2\,x}.$

23. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(c-a)(a-b)}+\dfrac{c+a}{(a-b)(b-c)}.$

24. $\dfrac{x+3y}{4(x+y)(y+2y)}+\dfrac{x+2y}{(x+y)(x+3y)}-\dfrac{x+y}{4(x+2y)(x+3y)}$.

25. $\dfrac{bc}{(c-a)(a-b)}+\dfrac{ac}{(a-b)(b-c)}+\dfrac{ab}{(b-c)(c-a)}$.

26. $\dfrac{5(2x-3)}{11(6x^2+x-1)}+\dfrac{7x}{6x^2+7x-3}-\dfrac{12(3x+1)}{11(4x^2+8x+3)}$.

27. $\dfrac{x}{x^3+y^3}-\dfrac{y}{x^3-y^3}+\dfrac{x^3y+xy^3}{x^6-y^6}$; $\dfrac{x^3+y^3}{x^2-xy+y^2}-\dfrac{x^3-y^3}{x^2+xy+y^2}$.

28. $\dfrac{2x+14}{x^3+x^2-49x-49}-\dfrac{1-x}{x^2-6x-7}$.

29. $\dfrac{x+c}{x^2-(a+b)x+ab}+\dfrac{x+b}{x^2-(a+c)x+ac}+\dfrac{x+a}{x^2-(b+c)x+bc}$.

30. $\dfrac{b}{(a-b)(a-c)}+\dfrac{a}{(b-a)(b-c)}$.

Suggestion. In finding the L. C. D. it is better to arrange the letters alphabetically. Thus,

$$\frac{b}{(a-b)(a-c)}+\frac{a}{(b-a)(b-c)}=\frac{b}{(a-b)(a-c)}+\frac{a\times -1}{[(b-a)\times -1](b-c)}=\text{etc.}$$

31. $\dfrac{x^2+2x+4}{x+2}-\dfrac{x^2-2x+4}{2-x}$; $\dfrac{x-2a}{x+a}+\dfrac{2(a^2-4ax)}{a^2-x^2}-\dfrac{3a}{x-a}$.

32. $\dfrac{1}{(m-2)(x+2)}+\dfrac{1}{(2-m)(x+m)}$; $\dfrac{1}{2x+1}+\dfrac{1}{2x-1}+\dfrac{4x}{1-4x^2}$; $\dfrac{2}{x+4}-\dfrac{x-3}{x^2-4x+16}+\dfrac{x^2}{x^3+64}$.

33. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-a)(b-c)}+\dfrac{c^2}{(c-a)(c-b)}$.

80. EXAMPLE 1. Find the product of $\frac{a}{b}$ and $\frac{c}{d}$.

Solution. Let $\frac{a}{b} = x$, and $\frac{c}{d} = y$. Multiplying both members of the first equation by b and both members of the second by d (Art. 47, Axiom 3), we have $a = bx$, and $c = dy$. Multiplying these two equations together, we have $ac = bdxy$. Dividing both members of this equation by bd (Art. 47, Axiom 4), gives

$$\frac{ac}{bd} = xy. \quad \text{But } xy = \frac{a}{b} \times \frac{c}{d}.$$

Therefore, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. Hence, in general,

To Multiply a Fraction by a Fraction. Multiply the numerators together for the numerator of the product, and the denominators together for the denominator of the product.

Notes: 1. Similarly, we may demonstrate the method when more than two fractions are multiplied together; also, for fractions whose terms are negative, integral, or fractional.

2. Since an entire or mixed expression may be expressed in fractional form, the method above is applicable to all cases. Thus,

$$m \times \frac{a}{b} = \frac{m}{1} \times \frac{a}{b} = \frac{am}{b}; \ \frac{a}{b} \times \left(m + \frac{n}{c}\right) = \frac{a}{b} \times \left(\frac{m}{1} + \frac{n}{c}\right) = \frac{am}{b} + \frac{an}{bc}.$$

EXAMPLE 2. Find the product of $\frac{4x^2 - 16x + 15}{2x^2 + 3x + 1}$, $\frac{x^2 - 6x - 7}{2x^2 - 17x + 21}$, and $\frac{4x^2 - 1}{4x^2 - 20x + 25}$.

Process.

$$\frac{4x^2 - 16x + 15}{2x^2 + 3x + 1} \quad \frac{x^2 - 6x - 7}{2x^2 - 17x + 21} \times \frac{4x^2 - 1}{4x^2 - 20x + 25}$$

$$= \frac{(2x-3)(2x-5)}{(2x+1)(x+1)} \times \frac{(x-7)(x+1)}{(2x-3)(x-7)} \times \frac{(2x+1)(2x-1)}{(2x-5)(2x-5)}$$

$$= \frac{(2x-3)(2x-5)(x-7)(x+1)(2x+1)(2x-1)}{(2x+1)(x+1)(2x-3)(x-7)(2x-5)(2x-5)} = \frac{2x-1}{2x-5}.$$

Explanation. Factoring the numerators and denominators of the fractions, multiplying the numerators together for the numerator of the product, and the denominators together for the denominator of the product, we have the third expression. Reducing the third expression to its lowest terms, gives the result.

Notes: 3. Observe the importance of factoring the terms of the fractions first. Also, indicate the multiplication of the numerators and denominators, and divide both terms of the fraction by their H. C. F. before performing the multiplication.

4. If the factors are mixed expressions, sometimes it is better to change them to fractional forms before performing the multiplication. Thus,

$$\left(a+\frac{ab}{a-b}\right)\left(b-\frac{ab}{a+b}\right)=\frac{a^2}{a-b}\times\frac{b^2}{a+b}=\frac{a^2b^2}{a^2-b^2}.$$

Example 3. Find the product of $\frac{2x^2+3x}{4x^3}$ and $\frac{4x^2-6x}{12x+18}$.

Process.

$$\frac{2x^2+3x}{4x^3}\times\frac{4x^2-6x}{12x+18}=\frac{x(2x+3)}{4x^3}\times\frac{2x(2x-3)}{6(2x+3)}$$

$$=\frac{x(2x+3)\times 2x(2x-3)}{4x^3\times 6(2x+3)}=\frac{2x-3}{12x}.$$

Exercise 69.

Simplify:

1. $\frac{a^2}{bc}\times\frac{b^2}{ac}\times\frac{c^2}{ab};\quad \frac{3a^3}{4c^2}\times\frac{2b^4}{21a^2}\times\frac{7c^2}{5ab};\quad \frac{x^n}{y^m}\times\frac{x^m}{y^n}.$

2. $\frac{3ab^2}{4cd}\times\frac{3ac^2}{2bd}\times\frac{8ad^2}{9bc};\quad \frac{3c^{-2}x^3}{5a^5y^{-\frac{2}{3}}}\times\frac{20c^3x}{9a^{-3}y^{-\frac{1}{3}}}.$

3. $\frac{x+1}{x-1}\times\frac{x+2}{x^2-1}\times\frac{x-1}{(x+2)^2};\quad \frac{3x^2-x}{5}\times\frac{10}{2x^2-4x}.$

4. $\frac{x^2+3x+2}{x^2+9x+20}\times\frac{x^2+7x+12}{x^2+5x+6};\quad \frac{m^2-n^2}{m^3-m^2n}\times\frac{m^2n}{m^3+n^3}.$

5. $\dfrac{x^6 - y^6}{x^4 + 2x^2y^2 + y^4} \times \dfrac{x^2 + y^2}{x^2 - xy + y^2} \times \dfrac{x + y}{x^3 - y^3}$.

6. $\dfrac{am}{a + m} \times \left(\dfrac{m}{a} - \dfrac{a}{m}\right)$; $\dfrac{m^2 + mn}{m^2 + n^2} \times \left(\dfrac{m}{m - n} - \dfrac{n}{m + n}\right)$.

7. $\dfrac{m^2 + mn}{m^2 + n^2} \times \dfrac{m^3 - n^3}{mn(m + n)}$; $\dfrac{x^2 - (a + b)x + ab}{x^2 - (a + c)x + ac} \times \dfrac{x^2 - c^2}{x^2 - b^2}$.

8. $\dfrac{m^3 - n^3}{m^2 - mn + n^2} \times \dfrac{m^3 + n^3}{m^2 + mn + n^2} \times \left(1 + \dfrac{n}{m - n}\right)$.

9. $\left(\dfrac{x^2}{a^2} - \dfrac{x}{a} + 1\right)\left(\dfrac{x^2}{a^2} + \dfrac{x}{a} + 1\right)$. **Suggestion.**

$$\left[\left(\frac{x^2}{a^2} + 1\right) - \frac{x}{a}\right]\left[\left(\frac{x^2}{a^2} + 1\right) + \frac{x}{a}\right] = \left(\frac{x^2}{a^2} + 1\right)^2 - \left(\frac{x}{a}\right)^2 = \text{etc.}$$

10. $\left(\dfrac{x}{a} - \dfrac{a}{x} + \dfrac{y}{b} - \dfrac{b}{y}\right) \times \left(\dfrac{x}{a} - \dfrac{a}{x} - \dfrac{y}{b} + \dfrac{b}{y}\right)$.

11. $\left(\dfrac{a}{bc} - \dfrac{b}{ac} - \dfrac{c}{ab} - \dfrac{2}{a}\right) \times \left(1 - \dfrac{2c}{a + b + c}\right)$.

12. $\dfrac{(a + b)^2 - c^2}{a^2 + ab - ac} \times \dfrac{a}{(a + c)^2 - b^2} \times \dfrac{(a - b)^2 - c^2}{ab - b^2 - bc}$.

13. $\dfrac{x^{2n} - 2x^n - 15}{x^{2n} - 2x^n - 63} \times \dfrac{x^{2n} + 6x^n - 7}{x^{2n} + 3x^n - 40} \times \dfrac{x^{2n} - x^n - 72}{x^{2n} + 4x^n + 3}$.

14. $\left[\dfrac{x^{2n}}{x^{2n} - y^{2m}} - \dfrac{y^{2m}}{x^{2n} + y^{2m}}\right] \times \left[\dfrac{(x^{2n} - y^{2m})^2}{(x^{2n} - y^{2m})^2 + (x^{2n} + y^{2m})^2}\right]$.

81. EXAMPLE 1. Find the quotient of $\dfrac{a}{b}$ divided by $\dfrac{c}{d}$.

Solution. Let x represent the quotient. Then $\dfrac{a}{b} \div \dfrac{c}{d} = x$. Since the quotient multiplied by the divisor gives the dividend,

we have $x \times \frac{c}{d} = \frac{a}{b}$. Multiplying both members of the equation by $\frac{d}{c}$, we have $x \times \frac{c}{d} \times \frac{d}{c} = \frac{a}{b} \times \frac{d}{c}$, or $x = \frac{a}{b} \times \frac{d}{c}$.

Therefore, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$. Hence, in general,

To Divide a Fraction by a Fraction. Invert the divisor, and proceed as in multiplication.

Notes: 1. Since an entire or mixed expression may be written in fractional form, the above method is applicable to all cases. Thus,

$$c \div \frac{a}{b} = \frac{c}{1} \div \frac{a}{b} = \frac{c}{1} \times \frac{b}{a} = \frac{bc}{a}; \quad \frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \times \frac{1}{c} = \frac{a}{bc}.$$

2. It is usually better to change mixed expressions to fractional form before performing the division. Thus,

$$\left(a - \frac{ab}{a+b}\right) \div \left(b - \frac{ab}{a+b}\right) = \frac{a^2}{a+b} \div \frac{b^2}{a+b} = \frac{a^2}{a+b} \times \frac{a+b}{b^2} = \frac{a^2}{b^2}.$$

Example 2. Divide $\frac{x^2 - 14x - 15}{x^2 - 4x - 45}$ by $\frac{x^2 - 12x - 45}{x^2 - 6x - 27}$.

Process.

$$\begin{aligned}\frac{x^2 - 14x - 15}{x^2 - 4x - 45} \div \frac{x^2 - 12x - 45}{x^2 - 6x - 27} &= \frac{(x-15)(x+1)}{(x-9)(x+5)} \div \frac{(x-15)(x+3)}{(x-9)(x+3)} \\ &= \frac{(x-15)(x+1)}{(x-9)(x+5)} \times \frac{(x-9)}{(x-15)} \\ &= \frac{(x-15)(x+1)(x-9)}{(x-9)(x+5)(x-15)} = \frac{x+1}{x+5}.\end{aligned}$$

Example 3. Divide $\frac{x^2}{y^3} + \frac{1}{x}$ by $\frac{x}{y^2} - \frac{1}{y} + \frac{1}{x}$.

Process.

$$\begin{aligned}\left(\frac{x^2}{y^3} + \frac{1}{x}\right) \div \left(\frac{x}{y^2} - \frac{1}{y} + \frac{1}{x}\right) &= \frac{x^3 + y^3}{xy^3} \div \frac{x^2 - xy + y^2}{xy^2} \\ &= \frac{(x+y)(x^2 - xy + y^2)}{xy^3} \times \frac{xy^2}{x^2 - xy + y^2} \\ &= \frac{x+y}{y} = \frac{x}{y} + 1.\end{aligned}$$

Exercise 70.

Divide:

1. $\dfrac{a\,x^2\,y^3}{b^2\,c^2}$ by $\dfrac{x^3\,y^2}{a\,b^2\,c}$; $\dfrac{3\,a^2\,b^3\,c^4}{4\,x^2\,y^3\,z^5}$ by $\dfrac{4\,a^4\,b^4\,c^3}{3\,x^4\,y^5\,z^3}$.

2. $\dfrac{2\,a^2\,x^{\frac{4}{3}}\,y}{6\,b^3\,c^{\frac{1}{4}}\,d^2}$ by $\dfrac{a\,x^{\frac{1}{3}}\,y^2}{b^2\,c^{\frac{5}{4}}\,d^2\,m}$; $\dfrac{3\,m}{2\,n-2}$ by $\dfrac{2\,m}{n-1}$.

3. $\dfrac{6\,(a\,b-b^2)}{a\,(a+b)^2}$ by $\dfrac{2\,b^2}{a\,(a^2-b^2)}$; $\dfrac{x^3-y^3}{x^3+y^3}$ by $\dfrac{(x-y)^2}{(x+y)^3}$.

4. $\dfrac{x^2-y^2}{m^3+8}$ by $\dfrac{x-y}{m+2}$; $\dfrac{x^2+x\,y+y^2}{x^3+y^3}$ by $\dfrac{x^3-y^3}{x^2-x\,y+y^2}$.

5. $\dfrac{2\,x^2+13\,x+15}{4\,x^2-9}$ by $\dfrac{2\,x^2+11\,x+5}{4\,x^2-1}$.

6. $\dfrac{x^2+x\,y+y^2}{x^2-x\,y+y^2}$ by $\dfrac{x+y}{x-y}$; $\dfrac{m^2}{n^2}-\dfrac{n^2}{m^2}$ by $\dfrac{m}{n}+\dfrac{n}{m}$.

7. $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}$ by $\dfrac{x+y}{x-y}$; $\dfrac{x-y}{x+y}$ by $\dfrac{x}{y}-\dfrac{y}{x}$.

8. $\dfrac{x^2+(a+c)\,x+a\,c}{x^2+(b+c)\,x+b\,c}$ by $\dfrac{x^2-a^2}{x^2-b^2}$.

9. $\dfrac{a^2+b^2+2\,a\,b-c^2}{c^2-a^2-b^2+2\,a\,b}$ by $\dfrac{a+b+c}{b+c-a}$.

10. $x^3-\dfrac{1}{x^3}$ by $x-\dfrac{1}{x}$; $a^2-b^2-c^2+2\,b\,c$ by $\dfrac{a+b-c}{a+b+c}$.

11. $\dfrac{m^{15}-x^{18}}{n^6+x^6}$ by $\dfrac{m^5-x^6}{n^2+x^2}$; $\dfrac{x^{12}+1}{a^6-1}$ by $\dfrac{x^4+1}{a^3+a^2-a-1}$.

12. $\dfrac{x^{-\frac{4}{3}}-x^{-\frac{1}{3}}}{2\,x^{-3}}$ by $\dfrac{x^{-\frac{2}{3}}+x^{-\frac{1}{6}}}{4\,x^2\,(x^{-\frac{2}{3}}-x^{-\frac{1}{6}})}$.

Exercise 71.

Perform the operations indicated in the following and reduce the results to their simplest forms :

1. $\left(\frac{x^2-7x+6}{x^2+3x-4} \div \frac{x^2+6x}{x^3-8x^2}\right) \times \frac{x^2+10x+24}{x^2-14x+48}$.

Process. $\left(\frac{x^2-7x+6}{x^2+3x-4} \div \frac{x^2+6x}{x^3-8x^2}\right) \times \frac{x^2+10x+24}{x^2-14x+48}$

$$= \left[\frac{(x-6)(x-1)}{(x+4)(x-1)} \div \frac{x(x+6)}{x^2(x-8)}\right] \times \frac{(x+4)(x+6)}{(x-6)(x-8)}$$

$$= \left[\frac{x-6}{x+4} \times \frac{x(x-8)}{x+6}\right] \times \frac{(x+4)(x+6)}{(x-6)(x-8)}$$

$$= \frac{x(x-6)(x-8)(x+4)(x+6)}{(x+4)(x+6)(x-6)(x-8)} = x.$$

2. $\frac{a-1}{a+1} \times \frac{a+1}{a-1} \div \frac{a^2-1}{a+1}$; $\left(\frac{a^3-b^3}{a-b} - \frac{a^3+b^3}{a+b}\right) \div \frac{4ab}{a^2-b^2}$.

3. $\left(\frac{2x}{x+y} + \frac{y}{x-y} - \frac{y^2}{x^2-y^2}\right) \div \left(\frac{1}{x+y} + \frac{x}{x^2-y^2}\right)$.

4. $\frac{x^2-x-20}{x^2-25} \times \frac{x^2-x-2}{x^2+2x-8} \times \frac{x^2-36}{x^2-6x} \div \frac{x+1}{x^2+5x}$.

5. $\frac{x^4-y^4}{a^2b+ab} \times \frac{a+b}{(x+y)^2} \times \frac{x^2-3xy+2y^2}{x^2+y^2} \div \frac{(x-y)^2}{ab}$.

6. $\left(\frac{a+b}{a-b} - \frac{a-b}{a+b}\right) \div \left(\frac{a+b}{a-b} + \frac{a-b}{a+b}\right)$.

7. $\frac{max}{nby} \times \frac{a^2-x^2}{c^2-x^2} \times \frac{bc+bx}{a^2+ax} \times \frac{c-x}{a-x} \div \frac{mx}{ny}$.

8. $\dfrac{1}{x+y} \div \left[\dfrac{y}{2}\left(\dfrac{1}{x+y}+\dfrac{1}{x-y}\right) \times \dfrac{x^2-y^2}{x^2y+xy^2}\right].$

9. $\dfrac{x^2+x-2}{x^2-x-20} \times \dfrac{x^2+5x+4}{x^2-x} \div \left(\dfrac{x^2+3x+2}{x^2-2x-15} \times \dfrac{x+3}{x}\right).$

10. $\left(\dfrac{a^{\frac{2}{3}}-b^{\frac{2}{3}}}{a^{\frac{4}{3}}-b^{\frac{2}{3}}} \div \dfrac{a^{\frac{1}{3}}+b^{\frac{1}{3}}}{a^{\frac{2}{3}}-b^{\frac{1}{3}}}\right) \times \dfrac{a^{\frac{2}{3}}+b^{\frac{1}{3}}}{a^{\frac{1}{3}}-b^{\frac{1}{3}}}.$

11. $\left(\dfrac{ax-a^2}{bx-b^2} \div \dfrac{x^2-a^2}{x^2-b^2}\right) \div \dfrac{bx+b^2}{ax+a^2}$; $\dfrac{9a^2b^3}{8x^3y^n} \times \dfrac{10a^{n-1}x^{n+2}}{21b^{m+3}y^{m-n}} \times \dfrac{x}{a}$; $\dfrac{64m^2n^2-a^2}{x^2-4} \times \dfrac{(x-2)^2}{8mn+a} \div \dfrac{x^2-4}{(x+2)^2}.$

12. $\dfrac{2a^2b^3c}{3x^2y^3} \times \dfrac{a^m b^n c^n}{x^m y^n} \times \dfrac{6x^{m-1}y^{n-2}}{a^{m+1}b^{n+2}c^{n+3}} \div \dfrac{a^2b}{c^2x^{n+4}y^{m+5}}.$

13. $\left(\dfrac{x^n}{y^m} \times \dfrac{x^m}{y^n} \times \dfrac{x^n}{y^n} \times \dfrac{y^m}{x^m}\right) \div \left(\dfrac{x^{m+n}}{y^{m+n}} \times \dfrac{x^{n-m}}{y^{n-m}}\right).$

14. $\left(x^4-\dfrac{1}{x^4}\right) \div \left(x-\dfrac{1}{x}\right)$, by inspection.

15. $\left(\dfrac{a^2}{b^2}-2+\dfrac{b^2}{a^2}\right) \div \left(\dfrac{a}{b}-\dfrac{b}{a}\right)$, by inspection.

16. $\left[x^3-\dfrac{1}{x^3}-3\left(x-\dfrac{1}{x}\right)\right] \div \left(x-\dfrac{1}{x}\right)$, by inspection.

17. $\dfrac{a^{m-n}b^{n-p}c^{p-m}}{x^{n-p}y^{p-m}z^{m-n}} \times \dfrac{a^{n-p}b^{p-m}c^{m-n}}{x^{p-1}y^{m-2}z^{n-3}} \times \dfrac{x^n y^p z^m}{a^m b^n c^p}.$

18. $\dfrac{6a^2b^2}{m+n} \div \left[\dfrac{3a(m-n)}{7(c+x)} \div \left\{\dfrac{4(c-x)}{21ab^2} \div \dfrac{c^2-x^2}{4(m^2-n^2)}\right\}\right].$

82. A **Complex Fraction** is one having a fraction in its numerator or denominator, or both; as,

$$\frac{\frac{a^2}{m}}{n};\quad \frac{\frac{a}{3}-a^2}{n+\frac{m}{n}}.$$

EXAMPLE 1. Reduce $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ to its simplest form.

Solution. A complex fraction may be regarded as representing the quotient of the numerator divided by the denominator. Hence,

$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}.$$

EXAMPLE 2. Reduce $\dfrac{a-\frac{b}{c}}{m}$ to its simplest form.

Solution. Since the divisor is m, we have

$$\frac{a-\frac{b}{c}}{m}=\left(a-\frac{b}{c}\right)\div m=\frac{ac-b}{c}\div\frac{m}{1}=\frac{ac-b}{c}\times\frac{1}{m}=\frac{ac-b}{cm}.$$

EXAMPLE 3. Reduce $\dfrac{1}{\frac{m}{n}}$, $\dfrac{m}{\frac{1}{n}}$, and $\dfrac{\frac{1}{m}}{\frac{1}{n}}$ to their simplest forms.

Process.

$$\frac{1}{\frac{m}{n}}=1\div\frac{m}{n}=1\times\frac{n}{m}=\frac{n}{m}.$$

$$\frac{m}{\frac{1}{n}}=m\div\frac{1}{n}=m\times\frac{n}{1}=mn.$$

$$\frac{\frac{1}{m}}{\frac{1}{n}}=\frac{1}{m}\div\frac{1}{n}=\frac{1}{m}\times\frac{n}{1}=\frac{n}{m}.$$ Hence, in general.

To Simplify a Complex Fraction. Divide the numerator by the denominator.

EXAMPLE 4. Simplify $\dfrac{\dfrac{m^2+n^2}{m^2-n^2}-\dfrac{m^2-n^2}{m^2+n^2}}{\dfrac{m+n}{m-n}-\dfrac{m-n}{m+n}}$.

Process.

$$\frac{\dfrac{m^2+n^2}{m^2-n^2}-\dfrac{m^2-n^2}{m^2+n^2}}{\dfrac{m+n}{m-n}-\dfrac{m-n}{m+n}}=\frac{\dfrac{(m^2+n^2)^2-(m^2-n^2)^2}{(m^2-n^2)(m^2+n^2)}}{\dfrac{(m+n)^2-(m-n)^2}{(m-n)(m+n)}}=\frac{\dfrac{4m^2n^2}{(m^2-n^2)(m^2+n^2)}}{\dfrac{4mn}{(m-n)(m+n)}}$$

$$=\frac{4m^2n^2}{(m^2-n^2)(m^2+n^2)}\times\frac{(m-n)(m+n)}{4mn}=\frac{mn}{m^2+n^2}.$$

EXAMPLE 5. Simplify $\dfrac{\dfrac{x}{x-y}-\dfrac{x}{x+y}}{\dfrac{y}{x-y}-\dfrac{x}{x+y}}$.

Solution. Multiplying both terms by $(x-y)(x+y)$, the L.C.D. of their denominators, we have

$$\frac{\left(\dfrac{x}{x-y}-\dfrac{x}{x+y}\right)[(x-y)(x+y)]}{\left(\dfrac{y}{x-y}+\dfrac{x}{x+y}\right)[(x-y)(x+y)]}=\frac{x(x+y)-x(x-y)}{y(x+y)+x(x-y)}=\frac{2xy}{x^2+y^2}.$$

Notes: 1. In many examples it is advisable to multiply both terms of the fraction by the L. C. D. of its denominators at once.

2. If the terms of the complex fraction are complicated, the beginner is advised to simplify each separately.

EXAMPLE 6. Simplify $\dfrac{\dfrac{mn}{x^2+(m+n)x+mn}-\dfrac{mp}{x^2+(m+p)x+mp}}{\dfrac{n-p}{x^2+(n+p)x+np}}$.

Process.

$$\frac{\dfrac{mn}{x^2+(m+n)x+mn}-\dfrac{mp}{x^2+(m+p)x+mp}}{\dfrac{n-p}{x^2+(n+p)x+np}}=\frac{\dfrac{mn}{(x+m)(x+n)}-\dfrac{mp}{(x+m)(x+p)}}{\dfrac{n-p}{(x+n)(x+p)}}$$

$$=\frac{\dfrac{mn(x+p)-mp(x+n)}{(x+m)(x+n)(x+p)}}{\dfrac{n-p}{(x+n)(x+p)}}=\frac{mx(n-p)}{(x+m)(x+n)(x+p)}\times\frac{(x+n)(x+p)}{n-p}$$

$$=\frac{mx}{x+m}.$$

EXAMPLE 7. Simplify $\dfrac{x}{1-\dfrac{1+x^2}{1-x+x^2+\dfrac{x}{1+x-\dfrac{x^2-1}{x}}}}$.

Solution. Begin with the complex fraction $\dfrac{x}{1+x-\dfrac{x^2-1}{x}}$. Thus,

$1+x-\dfrac{x^2-1}{x}=\dfrac{x+1}{x}$, and $\dfrac{x}{1+x-\dfrac{x^2-1}{x}}=\dfrac{x^2}{x+1}$. Similarly

$\dfrac{1+x^2}{1-x+x^2+\dfrac{x^2}{x+1}}=\dfrac{(1+x^2)(1+x)}{x^3+x^2+1}$, and $1-\dfrac{1+x^2}{1-x+x^2+\dfrac{x^2}{x+1}}$

$=-\dfrac{x}{x^3+x^2+1}$.

Therefore, $\dfrac{x}{1-\dfrac{1+x^2}{1-x+x^2+\dfrac{x}{1+x-\dfrac{x^2-1}{x}}}}=\dfrac{x}{-\dfrac{x}{x^3+x^2+1}}$

$=-(x^3+x^2+1)$.

Notes: 3. A fraction of the form in Example 7 is called a **Continued Fraction.**

4. To simplify a continued fraction, the student should always begin with the last complex fraction in the denominator.

Exercise 72.

Reduce to their simplest forms:

1. $\dfrac{x+6+\dfrac{1}{x-6}}{x-6+\dfrac{1}{x+6}}$; $\dfrac{1+\dfrac{a}{m}}{\dfrac{b}{m}-1}$; $\dfrac{x+\dfrac{a}{c}}{x+\dfrac{m}{n}}$; $\dfrac{xy-\dfrac{3x}{mn}}{\dfrac{mn}{x}+2n}$.

2. $\dfrac{a+b+\dfrac{b^2}{a}}{a+b+\dfrac{a^2}{b}}$; $\dfrac{\dfrac{1}{n}+\dfrac{1}{m}}{\dfrac{1}{n}-\dfrac{1}{m}}$; $\dfrac{\dfrac{m}{n}-\dfrac{b}{m}}{\dfrac{a}{m}+\dfrac{b}{n}}$; $\dfrac{\dfrac{m+n}{4mn}}{\dfrac{m^2-n^2}{8m^2n}}$.

3. $\dfrac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}}$; $\dfrac{\dfrac{x^2-17x+72}{x^2+22x+120}}{\dfrac{x^2-21x+108}{x^2+18x+80}}$.

4. $\dfrac{a+\dfrac{b-a}{1+ab}}{1-\dfrac{ab-a^2}{1+ab}}$; $\dfrac{1+\dfrac{(a-b)^2}{4ab}}{1-\dfrac{b^2+a^2}{2ab}}$; $\dfrac{\left(\dfrac{a}{x}+\dfrac{x}{a}\right)\left(\dfrac{a}{x}-\dfrac{x}{a}\right)}{1-\dfrac{x-a}{x+a}}$.

5. $\dfrac{\dfrac{1}{mn}+\dfrac{1}{mp}+\dfrac{1}{np}}{\dfrac{m^2-(n+p)^2}{mn}}$; $\left(\dfrac{x}{1+x}+\dfrac{1-x}{x}\right)\div\left(\dfrac{x}{1+x}-\dfrac{1-x}{x}\right)$.

6. $\dfrac{1}{x+\dfrac{1}{x+\dfrac{1}{x}}}$; $\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{x}}}$; $\dfrac{1}{x-\dfrac{x^2-1}{x+\dfrac{1}{x-1}}}$.

7. $\dfrac{1}{1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}}$; $\dfrac{x-2}{x-2-\dfrac{x}{x-\dfrac{x-1}{x-2}}}$; $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$.

8. $\dfrac{\dfrac{x^2}{y^3}+\dfrac{1}{x}}{\dfrac{x}{y^2}-\dfrac{1}{y}+\dfrac{1}{x}}$; $\dfrac{a}{b+\dfrac{c}{d+\dfrac{m}{n}}}$; $\dfrac{1-\dfrac{2xy}{(x+y)^2}}{1+\dfrac{2xy}{(x-y)^2}} \div \dfrac{1-\dfrac{y}{x}}{1+\dfrac{y}{x}}$.

9. $\dfrac{\dfrac{a}{x^2}+\dfrac{x}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{ax}+\dfrac{1}{x^2}}$; $\dfrac{\dfrac{x}{y}+\dfrac{y}{x}-1}{\dfrac{x^2}{y^2}+\dfrac{x}{y}+1} \times \dfrac{1+\dfrac{y}{x}}{x-y} \div \dfrac{1+\dfrac{y^3}{x^3}}{\dfrac{x^2}{y}-\dfrac{y^2}{x}}$.

10. $\dfrac{\dfrac{m^2+n^2}{m^2-n^2}+\dfrac{2m}{m+n}\left[\dfrac{mn-m^2}{(m-n)^2}+\dfrac{m+n}{m-n}\right]}{m-n}$.

11. $\left[\dfrac{n+\dfrac{m-n}{1+mn}}{1-\dfrac{(m-n)n}{1+mn}}-\dfrac{m-\dfrac{m-n}{1-mn}}{1-\dfrac{(m-n)m}{1-mn}}\right] \div \left(\dfrac{m}{n}-\dfrac{n}{m}\right)$.

83. Example. Find the third power of $\dfrac{a^m}{b^n}$.

Solution. Since an exponent shows how many times an expression is taken as a factor, we have

$$\left(\frac{a^m}{b^n}\right)^3 = \frac{a^m}{b^n} \times \frac{a^m}{b^n} \times \frac{a^m}{b^n} = \frac{(a^m)^3}{(b^n)^3} = \frac{a^{3m}}{b^{3n}}. \quad \text{Hence,}$$

To Find any Power of a Fraction. Raise both terms of the fraction to the required power.

Exercise 73.

Expand, by inspection, the following:

1. $\left(\frac{a^3 b^2}{3}\right)^4$; $\left(\frac{a^2 b}{2 x^3 y^2}\right)^5$; $\left(-\frac{2 a^3 b}{x y}\right)^5$; $\left(-\frac{3 a b^2}{4 x^3}\right)^4$.

2. $\left(-\frac{a^2 b^3 c^4}{2}\right)^7$; $\left(\frac{-m^2}{n^{\frac{1}{5}}}\right)^5$; $\left[-\left(\frac{2}{3}\right)^4 \times \left(\frac{3}{2}\right)^5\right]^3$.

3. $\left(\frac{2 a^3 b^{\frac{1}{5}} x^{\frac{1}{3}}}{3 m^{\frac{1}{2}} n^{\frac{1}{4}} y^{\frac{1}{6}}}\right)^4$; $\left[\frac{(x+y)^2}{(x-y)^2}\right]^2$; $\left[\frac{m(x-y)}{n(x+y)}\right]^2$.

4. $\left[\frac{(x+y)(x-y)}{m+n}\right]^2$; $\left[\frac{(a-b)^{-2}(a-b)^3}{(2a+3b)^{\frac{1}{2}}}\right]^4$; $\left(\frac{a(a-5)^{\frac{1}{5}}}{x^2}\right)^{15}$

5. $\left[-\left(\frac{(a-b)^{\frac{1}{2}}}{a+b}\right)^2\right]^2$; $\left(\frac{2 x^{\frac{2}{3}} - 3 y^n}{a^{\frac{2}{3}} b^{-\frac{1}{3}}}\right)^3$; $\left[\left(\frac{x}{y}\right)^{\frac{1}{m}} \times \left(\frac{x}{y}\right)^{\frac{1}{n}}\right]^{mn}$.

6. $-\left(\frac{a^n b^2}{c^m}\right)^6$; $\left[\left(\frac{x}{y}\right)^{\frac{1}{m}} \times \left(\frac{y}{x}\right)^{\frac{1}{n}}\right]^{mn}$; $\left[\dfrac{x^{\frac{1}{3}} \times \dfrac{\frac{1}{m^{\frac{1}{3}}} + 1}{a^{\frac{2}{3}}}}{n^{\frac{2}{3}} \times \dfrac{1 - \frac{1}{m^{\frac{2}{3}}}}{a}}\right]^3$.

84. EXAMPLE 1. Find the rth root of $\frac{a^m}{b^n}$.

Solution. Since the rth power is found by taking the numerator and denominator r times as a factor, the rth root is found by taking the rth root of each of its terms. The operation is indicated by dividing the exponent of each term by r. Thus,

$$\sqrt[r]{\frac{a^m}{b^n}} = \frac{a^{m \div r}}{b^{n \div r}} = \frac{a^{\frac{m}{r}}}{b^{\frac{n}{r}}}.$$

Illustration. $\sqrt[5]{\dfrac{32\,a^{10}}{243\,b^{15}}} = \dfrac{2^{5 \div 5}\,a^{10 \div 5}}{3^{5 \div 5}\,b^{15 \div 5}} = \dfrac{2\,a^2}{3\,b^3}$. Hence,

To Find any Root of a Fraction. Take the required root of each of its terms.

EXAMPLE 2. Find the square root of $\dfrac{x^2}{a^2} - 2 - ax + \dfrac{a^4}{4} + \dfrac{a^3}{x} + \dfrac{a^2}{x^2}$.

Arranging according to powers of a, we have

$$\left(\frac{a^2}{2} + \frac{a}{x} - \frac{x}{a}\right.$$

Process. $\dfrac{a^4}{4} + \dfrac{a^3}{x} + \dfrac{a^2}{x^2} - ax - 2 + \dfrac{x^2}{a^2}$

First term of the root squared, $\dfrac{a^4}{4}$

First remainder, $\dfrac{a^3}{x} + \dfrac{a^2}{x^2} - ax - 2 + \dfrac{x^2}{a^2}$

First trial divisor, a^2

First complete divisor, $a^2 + \dfrac{a}{x}$

$\dfrac{a}{x}$ times first complete divisor, $\dfrac{a^3}{x} + \dfrac{a^2}{x^2}$

Second remainder, $- ax - 2 + \dfrac{x^2}{a^2}$

Second trial divisor, $a^2 + \dfrac{2a}{x}$

Second complete divisor, $a^2 + \dfrac{2a}{x} - \dfrac{x}{a}$

$-\dfrac{x}{a}$ times second complete divisor, $- ax - 2 + \dfrac{x^2}{a^2}$

Note. If we take $-\dfrac{a^2}{2}$ for the square root of $\dfrac{a^4}{4}$, we shall arrive at the result $-\dfrac{a^2}{2} - \dfrac{a^2}{x} + \dfrac{x}{a}$.

Exercise 74.

Find the values of the following expressions:

1. $\sqrt[9]{\dfrac{x^{18}}{y^{27}z^{36}}}$; $\sqrt[4]{\dfrac{81\,m^8 n^{12}}{x^4}}$; $\sqrt[3]{-\dfrac{216\,a^3 c^{15}}{343\,x^{24}}}$; $\left(\dfrac{a^{10m}}{243\,b^{25n}}\right)^{\frac{1}{5}}$.

2. $\left(\dfrac{m^{50}n^{30}}{a^{100}}\right)^{\frac{1}{10}}$; $\left(-\dfrac{32\,a^{15}}{b^{10}}\right)^{\frac{1}{5}}$; $\left(-\dfrac{64\,m^3 n^6 x^9}{125\,a^6 b^{12} y^{15}}\right)^{\frac{1}{3}}$.

3. $\left(\dfrac{32\,a^{15}x}{243\,y^{-10}}\right)^{\frac{1}{5}}$; $\sqrt[n]{\sqrt{\left(\dfrac{a^{-2n}}{b^{\frac{n}{3}}}\right)^3}}$; $\sqrt[2p]{\dfrac{a^p x^{3p}}{y^{mp}z^{2p}}}$.

4. $\left(\dfrac{(x+y)^{2m}}{4^m(x-y)^{mn}}\right)^{\frac{1}{2m}}$; $\left\{\left[\left(\dfrac{a}{b}\right)^{\frac{m}{n}}\times\left(\dfrac{a}{b}\right)^{-\frac{n}{m}}\right]^{\frac{1}{m-n}}\right\}^{mn}$.

Find the square roots of:

5. $x^2+2x-1-\dfrac{2}{x}+\dfrac{1}{x^2}$; $a^4-a^3+\dfrac{a^2}{4}+4a-2+\dfrac{4}{a^2}$.

6. $\dfrac{a^2}{b^2}-\dfrac{2a}{b}+3-\dfrac{2b}{a}+\dfrac{b^2}{a^2}$; $\dfrac{15\,y}{x^2}-\dfrac{6\,y^{\frac{3}{2}}}{x}+y^2-\dfrac{18\,y^{\frac{1}{2}}}{x^3}+\dfrac{9}{x^4}$.

Miscellaneous Exercise 75.

Reduce to lowest terms:

1. $\dfrac{b(b-ax)+a(a+bx)}{(b-ax)^2+(a+bx)^2}$; $\dfrac{x^4-9x^3+7x^2+9x-8}{x^4+7x^3-9x^2-7x+8}$.

2. $\dfrac{21\,x^2y^2-35\,y^3z-12\,x^3z+20\,xyz^2}{18\,x^2z^2-21\,x^3y-30\,yz^3+35\,xy^2z}$.

3. $\dfrac{40\,x^2y^4-32\,x^3yz^2-5\,y^3z^3+4\,xz^5}{4\,xy^2z^3-36\,x^4z^2-5\,y^5z+45\,x^3y^3}$.

4. $\dfrac{x^3-6x^2-37x+210}{x^3+4x^2-47x-210}$; $\dfrac{x^{4n}+10x^{3n}+35x^{2n}+50x^n+24}{x^{3n}+9x^{2n}+26x^n+24}$.

Find the values of:

5. $3x^2+\dfrac{2xy^2}{z}-\dfrac{z^3}{y^2}$ when $x=4$, $y=\frac{1}{2}$, $z=1$; $\dfrac{a-x}{b-x}$ when $x=\dfrac{ab}{a+b}$.

6. $\dfrac{x^2+y^2-z^2+2xy}{x^2-y^2-z^2+2yz}$ when $x=4$, $y=\frac{1}{2}$, $z=1$.

7. $\dfrac{x-a}{b}-\dfrac{x-b}{a}$ when $x=\dfrac{a^2}{a-b}$; $\dfrac{x}{a}+\dfrac{x}{b-a}-\dfrac{a}{a+b}$ when $x=\dfrac{a^2(b-a)}{b(b+a)}$.

8. $\left(\dfrac{x-a}{x-b}\right)^3-\dfrac{x-2a+b}{x+a-2b}$ when $x=\dfrac{a+b}{2}$.

9. $\sqrt[3]{x\left[y^3+\sqrt[4]{\dfrac{x^2+2xy+y^2}{y^2}}\right]}$ when $x=8$, $y=1$.

10. $\dfrac{a^3-b^3}{2ab+2bc+2cd+2ad}\div\dfrac{a^2+b^2+c^2+d^2}{\sqrt{(a-c)(b+c)+6(-c-b)(a-2b)}}$ when $a=3$, $b=1$, $c=-2$, $d=0$.

11. $\dfrac{a^2+ac+b^2}{a^2-ac+b^2}-\dfrac{(4ab+b^2+d)^{\frac{1}{2}}}{\sqrt{4ab-b^2-2d}}-\dfrac{c}{a+b+c+d}$ when $a=4$, $b=3$, $c=1$, $d=7$.

Divide:

12. $\dfrac{m^3}{n^3}-x^3$ by $\dfrac{m}{n}-x$; $\dfrac{x^3}{y^3}+\dfrac{y^3}{x^3}$ by $\dfrac{x}{y}+\dfrac{y}{x}$.

13. $m^4 - \frac{1}{n^4}$ by $m - \frac{1}{n}$; $\frac{x^4}{y^4} - \frac{m^4}{n^4}$ by $\frac{x}{y} + \frac{m}{n}$.

14. $x^5 + \frac{1}{x^5}$ by $x + \frac{1}{x}$; $x^3 - \frac{1}{x^3} + x - \frac{1}{x}$ by $x - \frac{1}{x}$.

15. $a^2 - b^2 - c^2 - 2\,b\,c$ by $\frac{a+b+c}{a+b-c}$.

16. $x^3 + \frac{1}{x^3} - 3\left(\frac{1}{x^2} - x^2\right) + 4\left(x + \frac{1}{x}\right)$ by $x + \frac{1}{x}$.

17. $\frac{a + (a\,b^2)^{\frac{1}{3}} - (a^2\,b)^{\frac{1}{3}}}{a+b}$ by $\frac{a^{\frac{1}{3}}}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}$.

Factor:

18. $\frac{m^2}{n^2} - \frac{n^2}{m^2}$; $a^3 + \frac{1}{a^3}$; $\frac{a^6}{x^6} - \frac{b^6}{y^6}$; $\frac{a^3}{b^3} + \frac{x^3}{y^3}$.

19. $1 - \left(\frac{a+b}{a-b}\right)^2$; $(x+y)^3 + \left(\frac{a+b}{x+y}\right)^3$; $x^{4n}y^{4n} - \frac{x^{4n}}{y^{4n}}$.

Simplify:

20. $3\,x - \{y + [2\,x - (y - z)]\} + \frac{1}{2} + \frac{2\,z^2 - \frac{1}{2}}{2\,z + 1}$.

21. $$\frac{\frac{x-1}{3} + \frac{x-1}{x-2}}{\frac{x+2}{4} + \frac{x+2}{x-3}} \div \frac{\frac{x+3}{7} - \frac{x+3}{x+4}}{\frac{x-2}{3} + \frac{x-2}{x-1}}.$$

22. $$\frac{3\,a\,b\,c}{b\,c + a\,c - a\,b} - \frac{\frac{a-1}{a} + \frac{b-1}{b} + \frac{c-1}{c}}{\frac{1}{a} + \frac{1}{b} - \frac{1}{c}}.$$

23. $\dfrac{b}{(a-b)(a-c)} + \dfrac{a}{(b-a)(b-c)} + \dfrac{a+b}{(c-a)(c-b)}.$

24. $\dfrac{(x^{2m}-y^{2n})(2x^{2m}-2x^m y^n)}{4(x^m-y^n)^2 \div \dfrac{x^m y^n}{{}^m + y^n}}; \quad \dfrac{x+\dfrac{1}{y}}{x+\dfrac{1}{y+\dfrac{1}{z}}} - \dfrac{1}{y(xyz+x+z)}.$

25. $\dfrac{9b^2-(4c-2a)^2}{(2a+3b)^2-16c^2} + \dfrac{16c^2-(2a-3b)^2}{(3b+4c)^2-4a^2} + \dfrac{4a^2-(3b-4c)^2}{(2a+4c)^2-9b^2}.$

26. $\left(\dfrac{m^2}{n^2}-1\right)\left(\dfrac{m}{m-n}-1\right)+\left(\dfrac{m^3}{n^3}-1\right)\left(\dfrac{m^2+mn}{m^2+mn+n^2}-1\right).$

27. $\dfrac{\dfrac{1}{x}-\dfrac{x+a}{x^2+a^2}}{\dfrac{1}{a}-\dfrac{a+x}{a^2+x^2}} + \dfrac{\dfrac{1}{x}-\dfrac{x-a}{x^2+a^2}}{\dfrac{1}{a}-\dfrac{a-x}{a^2+x^2}}; \quad \dfrac{a+\dfrac{b}{1+\dfrac{a}{b}}}{a-\dfrac{b}{1-\dfrac{a}{b}}} \times (a^6-b^6).$

28. $\dfrac{a-b-c}{a^2-ac-ab+bc} + \dfrac{b-c-a}{b^2-ab-cb+ac} + \dfrac{c-a-b}{c^2-bc-ac+ab}.$

29. $\left(\dfrac{x}{1+\dfrac{1}{x}} + 1 - \dfrac{1}{x+1}\right) \div \left(\dfrac{x}{1-\dfrac{1}{x}} - x - \dfrac{1}{x-1}\right).$

30. $\dfrac{1}{x+\dfrac{1}{y+\dfrac{1}{z}}} \div \dfrac{1}{x+\dfrac{1}{y}} - \dfrac{1}{y(xyz+x+z)}.$

31. $$\frac{\left(\frac{3x+x^3}{1+3x^2}\right)^2-1}{\frac{3x^2-1}{x^3-3x}+1} \div \frac{\frac{9}{x^2}-\frac{33-x^2}{3x^2+1}}{\frac{3}{x^2}-\frac{2(x^2+3)}{(x^3-x)^2}};\quad \frac{\frac{3}{abc}}{\frac{1}{bc}+\frac{1}{ac}-\frac{1}{ab}}$$
$$-\frac{3-a-b+c}{a+b-c};\quad \frac{b^2}{a^8}\times\frac{a^{-1}b^{-2}}{ab^{-3}}\times\frac{a^2b^{-1}}{b^4}\div\left(\frac{a^{-2}b^{-1}}{a^2}\right)^2.$$

32. $$\frac{1+\frac{x^m-y^n}{x^m+y^n}}{1-\frac{x^m-y^n}{x^m+y^n}}\div\frac{1+\frac{x^{2m}-y^{2n}}{x^{2m}+y^{2n}}}{1-\frac{x^{2m}-y^{2n}}{x^{2m}+y^{2n}}};\quad \frac{a^{-\frac{1}{2}}b^{\frac{1}{2}}}{\frac{1}{b^{\frac{1}{2}}c^{-\frac{2}{3}}}}\times\frac{\frac{1}{b^{\frac{1}{3}}c^{\frac{1}{2}}}}{c^{\frac{1}{2}}a^{-\frac{1}{3}}}\times\frac{a^{\frac{1}{3}}c^3}{b^{\frac{1}{3}}c^{-\frac{2}{3}}}.$$

33. $$\frac{10xy-3y^2+10x-3y}{15y^2+10xy^2+30y+20xy}\div\frac{10x-3y}{45y+30xy}+\frac{3}{y+2}.$$

34. $$\left(\frac{4x^2}{y^2}-1\right)\left(\frac{2x}{2x-y}-1\right)+\left(\frac{8x^3}{y^3}-1\right)\left(\frac{4x^2+2xy}{4x^2+2xy+y^2}-1\right).$$

35. $$\frac{\frac{x^3-y^3}{x^2+y^2}\times\frac{x^2-y^2}{x^3+y^3}\times\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\div\left(\frac{1}{y}-\frac{1}{x}\right)}{\frac{(x+y)^2-xy}{(x-y)^2+xy}}.$$

36. $$\left[\frac{a^4-y^4}{a^2-2ay+y^2}\div\frac{a^2+ay}{a-y}\right]\times\left[\frac{a^5-a^3y^2}{a^3+y^3}\div\frac{a^4-2a^3y+a^2y^2}{a^2-ay+y^2}\right].$$

37. $$\frac{b}{3ax-5by}-\frac{1}{\frac{ax}{b}-y-\frac{2by^2}{3ax-2by}}.$$

38. $$\frac{1}{1+\frac{x}{y+z}}+\frac{1}{1+\frac{y}{x+z}}+\frac{1}{1+\frac{z}{x+y}}.$$

39. $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{(b+c)(a+c)(a+b)}{a\,b\,c}$.

40. $\dfrac{\left[\frac{(a+b)^2}{4ab}-1\right]\left[\frac{(a-b)^2}{4ab}+1\right]}{(a+b)^3-3a^2b-3ab^2}\times\dfrac{[(a+b)^2-ab][(a-b)^2+ab]}{(a-b)^3+3ab(a-b)}$.

41. $\frac{(b-c)^2}{(c-a)(a-b)}+\frac{(c-a)^2}{(a-b)(b-c)}+\frac{(a-b)^2}{(b-c)(c-a)}$.

42. $\left(\frac{x^3-y^3}{x^2-y^2}-\frac{10x^2-13xy-3y^2}{10x^2-3xy-y^2}-\frac{2x^3+xy^2+xy+y^2}{2x^2+xy-y^2}\right)$
$\div\frac{xy-y^2-2x+2y}{2x-y}$.

43. $\frac{(a+b)^3-c^3}{(a+b)-c}+\frac{(b+c)^3-a^3}{b+c-a}+\frac{(a+c)^3-b^3}{a+c-b}$.

44. $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}-\frac{1}{a\,b\,c}$.

45. $\frac{2a+n}{am+ab-bm-a^2}+\frac{a+b+n}{ab+bm-am-b^2}-\frac{m+n-a}{m^2-bm-am+ab}$.

46. $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}$.

Queries. Why does changing the sign of one factor of either term of a fraction change the sign of that term? Will it change the sign of the fraction? Why? When the denominators have no common factors why multiply both terms by the product of the denominators of all the other fractions? Why does the process of reducing to forms having a common denominator not change the value of a fraction? How prove the methods for addition, subtraction, multiplication, and division of fractions?

CHAPTER XV.

FRACTIONAL EQUATIONS.

85. Example 1. Solve $\frac{6-5x}{15} - \frac{7-2x^2}{14(x-1)} = \frac{1+3x}{21} - \frac{10x-11}{30} + \frac{1}{105}$.

Solution. Multiplying each member by 210 (the L. C. M. of 15, 21, 30, and 105), transposing and uniting like terms, we have $-\frac{105-30x^2}{x-1} = 5 + 30x$. Multiplying each member of this equation by $x-1$, transposing and uniting like terms, we have $25x = 100$. $\therefore\ x = 4$.

Proof. Substituting 4 for x in the given equation, we have

$$\frac{6-5\times 4}{15} - \frac{7-2\times 4^2}{14(4-1)} = \frac{1+3\times 4}{21} - \frac{10\times 4-11}{30} + \frac{1}{105},$$

or, $-\frac{71}{210} = -\frac{71}{210}$, which is an identity. Hence,

To Clear an Equation of Fractions. Multiply each member by the L. C. M. of the denominators.

Example 2. Solve $\frac{2x+1\frac{1}{2}}{5} - \frac{2\frac{2}{5}x-1}{50x-10} = \frac{x-\frac{1}{2}}{2\frac{1}{2}}$.

Process. Multiply by 5, $2x + 1\frac{1}{2} - \frac{2\frac{2}{5}x-1}{10x-2} = 2x-1$.

Transpose and unite, $-\frac{2\frac{2}{5}x-1}{10x-2} = -2\frac{1}{2}$.

Clear of fractions, $-(2\frac{2}{5}x-1) = -25x+5$.

Transpose and unite, $22.6x = 4$. $\therefore\ x = \frac{20}{113}$.

Note. In solving a fractional equation, where some of the denominators are simple and some are compound expressions, it is better to multiply each member of the equation by an expression which will remove the simple denominators first, then transpose (if necessary) and unite like terms. Similarly remove the compound denominators of the resulting equation.

Exercise 76.

Solve the following equations:

1. $\frac{12}{x}+\frac{1}{12x}=\frac{29}{24}$; $\frac{5x-1}{2}-\frac{7-3x}{3x}=\frac{10x-3}{4}-\frac{3-5x}{2x}$.

2. $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}$; $\frac{2x-5}{5}+\frac{x-3}{2x-15}=\frac{4x-3}{10}-1\frac{1}{10}$.

3. $\frac{9x+5}{14}+\frac{8x-7}{6x+2}=\frac{36x+15}{56}+\frac{10\frac{1}{4}}{14}$.

4. $\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}$; $\frac{3x+2}{6}-\frac{2x-1}{3x-7}=\frac{x}{2}$.

5. $\frac{18x-22}{39-6x}+2x+\frac{1+16x}{24}=4\frac{5}{12}-\frac{101-64x}{24}$.

6. $\frac{18x+10}{42}-\frac{72x+30}{168}=\frac{20.5}{42}-\frac{16x-14}{18x+6}$.

7. $\frac{1}{2}+\frac{2}{x+2}=\frac{x+2}{2x}$; $\frac{4(x+3)}{9}=\frac{8x+37}{18}-\frac{7x-29}{5x-12}$.

8. $\frac{2x+8\frac{1}{2}}{9}-\frac{13x-2}{17x-32}+\frac{x}{3}=\frac{7x}{12}-\frac{x+16}{36}$.

9. $\frac{6x+1}{15}-\frac{2x-4}{7x-16}=\frac{2x-1}{5}$; $\frac{x-2}{.05}-\frac{x-4}{.0625}=56$.

86. Frequently it is better to unite some of the terms before clearing the equation of fractions. Thus,

EXAMPLE 1. Solve $\dfrac{25-\frac{x}{3}}{x+1}+\dfrac{16\,x+4.2}{3\,x+2}=5+\dfrac{23}{x+1}.$

Process. Transpose, $\dfrac{25-\frac{x}{3}}{x+1}-\dfrac{23}{x+1}+\dfrac{16x+4.2}{3x+2}=5.$

Unite like terms, $\dfrac{2-\frac{x}{3}}{x+1}+\dfrac{16x+4.2}{3x+2}=5.$

Free from fractions, $4+\frac{16}{3}x-x^2+16\,x^2+20.2\,x+4.2=15\,x^2+25\,x+10.$

Transpose and unite, $\dfrac{1.6\,x}{3}=1.8.$

$\therefore\; x=3\frac{3}{8}.$

EXAMPLE 2. Solve $\dfrac{1}{x-2}-\dfrac{1}{x+2}-\dfrac{x+1}{x^2-4}=0.$

Process. Multiply by x^2-4, $(x+2)-(x-2)-(x+1)=0.$
Simplify, $-x+3=0. \quad \therefore\; x=3.$

Notes: 1. If a fraction is preceded by the $-$ sign, in clearing the equation of fractions, care must be taken to change the sign of each term of the numerator. In such case it is convenient to enclose the numerator in parentheses before clearing the equation of fractions.

2. The student should be careful to observe that he can make but two classes of changes upon an equation without destroying the equality:

I. **Such as do not affect the value of the members.**

II. **Such as affect both members equally.**

Thus, in the above process, the first operation affects both members equally; and the second, that of uniting like terms, does not affect the value of the members.

EXAMPLE 3. Solve $\dfrac{4}{x+3}-\dfrac{2}{x+1}=\dfrac{5}{2\,x+6}-\dfrac{2\frac{1}{2}}{2\,x+2}.$

Solution. Transposing, $\dfrac{4}{x+3}-\dfrac{5}{2\,x+6}=\dfrac{2}{x+1}-\dfrac{2\frac{1}{2}}{2\,x+2}$

Simplifying each member separately, we have

$$\frac{3}{2(x+3)} = \frac{1\frac{1}{2}}{2(x+1)}, \text{ or } \frac{1}{x+3} = \frac{1}{2(x+1)}.$$

Clearing of fractions, we have $2(x+1) = x+3$. $\therefore x = 1$.

Example 4. Solve $\dfrac{x-4}{x-5} - \dfrac{x-5}{x-6} = \dfrac{x-7}{x-8} - \dfrac{x-8}{x-9}$.

Solution. Reduce the fractions to mixed expressions,

$$\left(1+\frac{1}{x-5}\right) - \left(1+\frac{1}{x-6}\right) = \left(1+\frac{1}{x-8}\right) - \left(1+\frac{1}{x-9}\right),$$

or $\dfrac{1}{x-5} - \dfrac{1}{x-6} = \dfrac{1}{x-8} - \dfrac{1}{x-9}$. Reducing the terms in each member separately to common denominators and adding, we get $-\dfrac{1}{(x-5)(x-6)} = -\dfrac{1}{(x-8)(x-9)}$. Clearing this equation of fractions, we have $-(x-8)(x-9) = -(x-5)(x-6)$. Simplifying, transposing, and uniting like terms, $-6x = -42$. $\therefore x = 7$.

Example 5. Solve $\dfrac{(2x+3)x}{2x+1} + \dfrac{1}{3x} = x+1$.

Process. Reduce $\dfrac{(2x+3)x}{2x+1}$ to a mixed expression,

$$x+1-\frac{1}{2x+1}+\frac{1}{3x} = x+1.$$

Transpose and unite, $$-\frac{1}{2x+1} = -\frac{1}{3x}.$$

Clear of fractions, $$-3x = -2x-1.$$

Therefore, $$x = 1.$$

Example 6. Solve $\dfrac{5x-64}{x-13} - \dfrac{2x-11}{x-6} = \dfrac{4x-55}{x-14} - \dfrac{x-6}{x-7}$.

Process. Reduce the fractions to mixed expressions,

$$5+\frac{1}{x-13} - \left(2+\frac{1}{x-6}\right) = 4+\frac{1}{x-14} - \left(1+\frac{1}{x-7}\right).$$

Simplify each member separately,

$$\frac{7}{(x-13)(x-6)} = \frac{7}{(x-14)(x-7)}.$$

Divide by 7 and clear of fractions,

$$x^2 - 21x + 98 = x^2 - 19x + 78.$$

Therefore, $\quad x = 10.$

Exercise 77.

Solve the following equations:

1. $\dfrac{12}{x} + \dfrac{1}{12x} = \dfrac{29}{2.4}; \quad \dfrac{x+4}{3x-8} = \dfrac{x+5}{3x-7}.$

2. $\dfrac{3x+1}{3(x-2)} = \dfrac{x-2}{x-1}; \quad \dfrac{6x+1}{15} - \dfrac{2x-4}{7x-16} = \dfrac{2x-1}{5}.$

3. $\dfrac{x+25}{x-5} = \dfrac{2x+75}{2x-15}; \quad \dfrac{5}{1-5x} + \dfrac{4}{2x-1} = \dfrac{3}{3x-1}.$

4. $\dfrac{6x+8}{2x+1} - \dfrac{2x+38}{x+12} = 1; \quad \dfrac{x^2-x+1}{x-1} + \dfrac{x^2+x+1}{x+1} = 2x.$

5. $\dfrac{x-7}{x+7} - \dfrac{2x-15}{2x-6} + \dfrac{1}{2x+14} = 0; \quad \dfrac{3}{1-x} - \dfrac{2}{1+x} - \dfrac{1}{1-x^2} = 0.$

6. $\dfrac{3}{4-2x} + \dfrac{30}{8(1-x)} = \dfrac{3}{2-x} + \dfrac{5}{2-2x}.$

7. $\dfrac{6x-7\frac{1}{3}}{13-12x} + 2x + \dfrac{1+16x}{24} = 4\frac{5}{12} - \dfrac{12\frac{5}{8}-8x}{3}.$

8. $\dfrac{x-1}{x-2} + \dfrac{x-5}{x-6} = \dfrac{x-4}{x-5} + \dfrac{x-2}{x-3}.$

9. $\dfrac{5x-8}{x-2} + \dfrac{6x-44}{x-7} - \dfrac{10x-8}{x-1} = \dfrac{x-8}{x-6}.$

10. $\dfrac{x-1}{x-2}+\dfrac{x+1}{x+2}=\dfrac{2(x^2+4x+1)}{(x+2)^2}.$

11. $\dfrac{30+6x}{x+1}+\dfrac{60+8x}{x+3}=14+\dfrac{48}{x+1}.$

12. $\dfrac{.6x+.044}{.4}-\dfrac{.5x-.178}{.6}=.38;\ \dfrac{.3x-1}{.5x-.4}=\dfrac{.5+1.2x}{2x-.1}.$

13. $\dfrac{2x-3}{.3x-.4}=\dfrac{.4x-.6}{.06x-.07};\ \dfrac{1-1.4x}{x+.2}=\dfrac{.7(x-1)}{.1-.5x}.$

87. A **Literal Equation** is one in which some known number is represented by a letter; as,

EXAMPLE 1. Solve $\dfrac{x}{m}+\dfrac{x}{n-m}=\dfrac{m}{m+n}.$

Process. Clear of fractions, $x(n^2-m^2)+x(m^2+mn)=m^2(n-m).$

Unite like terms, $(n^2+mn)x=m^2(n-m).$

Divide by $n(n+m)$, $x=\dfrac{m^2(n-m)}{n(n+m)}.$

EXAMPLE 2. Solve $(x-m)(x-n)-(x-n)(x-a)=2(x-m)(m-a).$

Process. Simplify, transpose, and unite,

$$3ax-3mx=-2m^2+2am-mn+an.$$

Factor, $3(a-m)x=(a-m)(2m+n).$

Divide by $3(a-m)$, $x=\dfrac{2m+n}{3}.$

EXAMPLE 3. Solve $ax-\dfrac{a^2-3bx}{a}-ab^2=bx+\dfrac{6bx-5a^2}{2a}-\dfrac{bx+4a}{4}.$

Process. Clear of fractions, simplify, transpose, and unite,

$$4a^2x-3abx=4a^2b^2-10a^2.$$

Factor, $a(4a-3b)x=2a^2(2b^2-5).$

Divide by $a(4a-3b)$, $x=\dfrac{2a(2b^2-5)}{4a-3b}.$

EXAMPLE 4. Solve $\frac{ax-b}{ax+b}-\frac{bx-a}{bx+a}=\frac{a-b}{(ax+b)(bx+a)}$.

Solution. Reducing the terms of the first member to mixed expressions, we have $\left(1-\frac{2b}{ax+b}\right)-\left(1-\frac{2a}{bx+a}\right)=\frac{a-b}{(ax+b)(bx+a)}$. Uniting like terms and reducing the fractions to a common denominator, adding and factoring their numerators, we have $\frac{2(a+b)(a-b)x}{(ax+b)(bx+a)}=\frac{a-b}{(ax+b)(bx+a)}$. Clearing of fractions, $2(a+b)(a-b)x=a-b$. Therefore, $x=\frac{1}{2(a+b)}$.

Notes: 1. Example 4 may be solved by clearing the equation of fractions. The solution is presented as an expeditious method.

2. If the student cannot readily discover a special artifice, he should clear the equation of fractions at once.

3. Known terms are called **absolute** terms. Thus, in the equation $mx^2+nx+a=0$, a is called the absolute term.

EXAMPLE 5. Solve $\frac{a+b}{x-c}-\frac{a}{x-a}-\frac{b}{x-b}=0$.

Process. Clear of fractions,

$$(a+b)(x-a)(x-b)-a(x-b)(x-c)-b(x-a)(x-c)=0.$$

Simplify, transpose, and factor,

$$x(ac+bc-a^2-b^2)=ab(2c-a-b).$$

Divide by $ac+bc-a^2-b^2$,

$$x=\frac{ab(2c-a-b)}{ac+bc-a^2-b^2},$$

or

$$x=\frac{ab(a+b-2c)}{a^2+b^2-c(a+b)}.$$

Exercise 78.

Solve the following equations:

1. $2ax+m=5a-nx$; $\frac{a}{x}+\frac{b}{a}=\frac{1}{3x}$.

2. $10bmx-6an=2am-5bnx$; $\frac{1}{a}-\frac{1}{x}=\frac{1}{x}-\frac{1}{b}$.

3. $\frac{m^2}{x} + \frac{n}{2} = \frac{4n^2}{x} + \frac{m}{4}$; $\frac{a}{bx} + \frac{b}{ax} = a^2 + b^2$.

4. $\frac{1}{4}x(x-a) - \left(\frac{x+a}{2}\right)^2 = \frac{2a}{3}\left(x - \frac{a}{2}\right)$.

5. $\frac{x-a}{b} - \frac{x+b}{a} + 2 = 0$; $(x-a)(x-b) = (x-a-b)^2$.

6. $\frac{a(b^2x + x^3)}{bx} = acx + \frac{ax^2}{b}$; $\frac{2}{3}\left(\frac{x}{a} + 1\right) = \frac{3}{4}\left(\frac{x}{a} - 1\right)$.

7. $(x+n)^3 - (x-n)^3 - n(3x-n)(2x+n) = x(n+1) + 3$.

8. $\frac{3}{c} - \frac{ab - x^2}{bx} = \frac{4x - ac}{cx}$; $\frac{x^2 - a}{bx} - \frac{a - x}{b} = \frac{2x}{b} - \frac{a}{x}$.

9. $\frac{x-m}{m} + \frac{x^2 - mx - n^2}{mx - n^2} = 1 + \frac{n^2}{mx - n^2}$.

Miscellaneous Exercise 79.

Solve the equations :

1. $\frac{x}{9} = \frac{x+1}{3} - \frac{7 - 2x^2}{1 - 9x}$; $\frac{ac}{bx} - \frac{bc}{ax} = a + b$.

2. $\frac{ax + b}{ax - b} - \frac{3b}{ax + b} = \frac{a^2x^2 + b^2}{a^2x^2 - b^2}$.

3. $\frac{x}{x-2} - \frac{x+1}{x-1} = \frac{x-8}{x-6} - \frac{x-9}{x-7}$.

4. $1 - \frac{2(2x+3)}{63 - 9x} = \frac{6}{7 - x} - \frac{5x + 1}{28 - 4x}$.

5. $\dfrac{1}{a(b-x)}+\dfrac{1}{b(c-x)}-\dfrac{1}{a(c-x)}=0.$

6. $(2x-a)\left(x+\dfrac{2a}{3}\right)=4x\left(\dfrac{a}{3}-x\right)-\frac{1}{2}(a-4x)(2a+3x).$

7. $\dfrac{17}{x+3}-4=\dfrac{105+10x}{3x+9}-10.$

8. $(x+3)^2-\dfrac{8x(x+5)}{10}=7x-\left(3x-\dfrac{x(x-5\frac{1}{2})}{5}\right).$

9. $\dfrac{x-a}{a-b}-\dfrac{a+x}{a+b}-\dfrac{2ax}{a^2-b^2}=0;\quad \dfrac{1}{x-a}-\dfrac{1}{x-b}=\dfrac{a-b}{x^2-ab}.$

10. $\dfrac{x^2-x+1}{x-1}+\dfrac{x^2+x+1}{x+1}=2x;\quad \dfrac{a}{x}=c(a-b)+\dfrac{b}{x}.$

11. $\dfrac{x+2}{x}+\dfrac{x-7}{x-5}-\dfrac{x+3}{x+1}=\dfrac{x-6}{x-4}.$

12. $.15x+\dfrac{.135x-.225}{.6}=\dfrac{.36}{.2}-\dfrac{.09x-.18}{.9}.$

13. $\dfrac{x-a}{x-a-1}-\dfrac{x-a-1}{x-a-2}=\dfrac{x-b}{x-b-1}-\dfrac{x-b-1}{x-b-2}.$

14. $\dfrac{30a-bx}{5a}-\dfrac{9n-ax}{3n}-\dfrac{6m-nx}{2m}=0.$

15. $\dfrac{4m(a^2-5x^2)}{8x}=7mn+\dfrac{5m(b^2-2x)}{4}.$

16. $p-\dfrac{x-np}{m}=\dfrac{x-mp}{n}-\dfrac{x-mn}{p}-p.$

17. $$\frac{x-\frac{2(x-18)}{9}}{8}-\frac{x-18}{6}=x+9-\frac{5x-\frac{2(x+10)}{23}}{4}.$$

18. $$\frac{3b(x-a)}{5a}+\frac{x-b^2}{15b}=-\frac{b(4a+cx)}{6a}.$$

19. $$\frac{c^2-3dx}{c^2+3dx}+\frac{d^2+2cx}{d^2-2cx}=2;\quad \frac{\frac{mx-n}{x}}{\frac{m}{n}}=\frac{\frac{mx+n}{x}}{\frac{n}{m}}.$$

20. $$\frac{m}{x}+\frac{x}{m}+\frac{m(x-m)}{x(x+m)}-\frac{x(x+m)}{m(x-m)}=\frac{mx}{m^2-x^2}-2.$$

Queries. Upon what principle is an equation cleared of fractions? How is it done? Why change the signs of the terms of the numerator of a fraction, preceded by a minus sign, when clearing of fractions? Upon what principle (give four different explanations) may the signs of all the terms of an equation be changed?

Exercise 80.

1. The second digit of a number exceeds the first by 3; and if the number, increased by 36, be divided by the sum of its digits, the quotient is 10. Find the number.

Solution. Let $x =$ the digit in tens' place.

Then $x + 3 =$ the digit in units' place,

and $2x + 3 =$ the sum of the digits.

Therefore, $10x + x + 3$, or $11x + 3 =$ the number.

Hence, $\frac{11x+3+36}{2x+3}=10.$ $\therefore x=1.$ $11x+3=14$, the number.

2. The first digit of a number is three times the second; and if the number, increased by 3, be divided by the difference of the digits, the quotient is 17. Find the number.

3. The first digit of a number exceeds the second by 4; and if the number be divided by the sum of its digits, the quotient is 7. Find the number.

4. The second digit of a number exceeds the first by 3; and if the number, diminished by 9, be divided by the sum of its digits, the quotient is 3. Find the number.

5. A can do a piece of work in 7 days, and B can do it in 5 days. How long will it take A and B together to do the work?

Solution. Let $x =$ the *number* of days it will take A and B together.

Then $\frac{1}{x} =$ the *part* they do in one day;

but $\frac{1}{7} =$ the *part* A can do in one day,

and $\frac{1}{5} =$ the *part* B can do in one day.

Therefore, $\frac{1}{7} + \frac{1}{5} =$ the *part* A and B can do in one day.

Hence, $\frac{1}{x} = \frac{1}{7} + \frac{1}{5}$. Therefore, $x = 2\frac{11}{12}$.

6. A can do a piece of work in $2\frac{1}{2}$ days, B in 3 days, and C in 5 days. In what time will they do it, all working together?

7. A can do a piece of work in a days, B in b days, C in c days. In what time will they do it, all working together?

8. A and B together can do a piece of work in 12 days, A and C in 15 days, B and C in 20 days. In what time can they do it, all working together?

9. A and B together can do a piece of work in a days, A and C in b days, B and C in c days. In what time can they do it, all working together? In what time can each do it alone?

10. A tank can be emptied by three pipes in 80 minutes, 200 minutes, and 5 hours, respectively. In what time will it be emptied if all three are running together?

11. A sets out and travels at the rate of 9 miles in 5 hours. Six hours afterwards, B sets out from the same place and travels in the same direction, at the rate of 11 miles in 6 hours. In how many hours will he overtake A?

Solution. Let $x =$ the *number* of hours B travels.

Then $x + 6 =$ the *number* of hours A travels;

also, $\frac{9}{5} =$ the *number* of miles per hour A travels,

and $\frac{11}{6} =$ the *number* of miles per hour B travels.

Then, $\frac{11}{6}x =$ the *number* of miles B travels,

and $\frac{9}{5}(x + 6) =$ the *number* of miles A travels.

Hence, $\frac{11}{6}x = \frac{9}{5}(x + 6)$. Therefore, $x = 324$.

12. A man walked to the top of a mountain at the rate of 2 miles an hour, and down the same way at the rate of $3\frac{1}{5}$ miles an hour, and is out 13 hours. How far is it to the top of the mountain?

13. A person has a hours at his disposal. How far may he ride in a coach which travels b miles an hour, so as to return home in time, if he can walk at the rate of c miles an hour?

14. In going a certain distance, a train travelling 55 miles an hour takes 3 hours less than one travelling 45 miles an hour. Find the distance.

15. The distance between London and Edinburgh is 360 miles. One traveller starts from London and travels at the rate of 5 miles an hour; another starts at the same time from Edinburgh, and travels at the rate of 7 miles an hour. How far from London will they meet?

16. The distance between A and B is 154 miles. One traveller starts from A and travels at the rate of 3 miles in 2 hours; another starts at the same time from B, and travels at the rate of 5 miles in 4 hours. How long and how far did each travel before they met?

17. The distance between A and B is a miles. One traveller starts from A and travels at the rate of m miles in n hours; another starts at the same time from B, and travels at the rate of b miles in c hours. How long and how far did each travel before they met?

18. If it takes m pieces of one kind of money to make a dollar, and n pieces of another kind to make a dollar, how many pieces of each kind will it take to make one dollar containing c pieces?

19. The denominator of a certain fraction exceeds the numerator by 6; and if 8 be added to the denominator, the value of the fraction is $\frac{1}{3}$. Find the fraction.

20. A can do a piece of work in $2m$ days, B and A together in n days, and A and C in $m + \frac{n}{2}$ days. In what time will they do it, all working together?

21. In a composition of a certain number of pounds of gunpowder the nitre was 10 pounds more than $\frac{2}{3}$ of the whole, the sulphur was $4\frac{1}{2}$ pounds less than $\frac{1}{6}$ of the whole, and the charcoal 2 pounds less than $\frac{1}{7}$ of the nitre. Find the number of pounds in the gunpowder.

22. A broker invests $\frac{3}{8}$ of a certain sum in 5 % bonds, and the remainder in 6 bonds; his annual income is $180. Find the amount in each kind of bond, and the sum.

23. A broker invests $\frac{m}{n}$th of a certain sum in a % bonds, and the remainder in c % bonds; his annual income is b dollars. Find the amount in each kind of bond, and the sum invested.

24. At the same time that the west-bound train going at the rate of 33 miles an hour passed A, the east-bound train going at the rate of 21 miles an hour passed B; they collided 18 miles beyond the midway station from A. How far is A from B?

25. A person setting out on a journey drove at the rate of a miles an hour to the nearest railway station, distant b miles from his home. On arriving at the station he found that the train had left c hours before. At what rate should he have driven in order to reach the station just in time for the train?

26. A merchant drew every year, upon the money he had in business, the sum of a dollars for expenses. His profits each year were the nth part of what remained after this deduction, but at the end 3 years he found his money exhausted. How many dollars had he in the beginning?

CHAPTER XVI.

SIMULTANEOUS SIMPLE EQUATIONS.

88. Simultaneous Equations are such as are satisfied by the same values of the unknown numbers.

Thus, $3x + y = 9$ and $5x - 2y = 4$ are satisfied only by $x = 2$ and $y = 3$.

Elimination is the process of combining simultaneous equations so as to cause one or more of the unknown numbers to disappear.

This process enables us to form an equation containing but *one* unknown number. The equation thus formed can be solved as shown in the preceding chapter.

Note. There are only three methods of elimination most commonly used.

Elimination by Addition or Subtraction.

89. Example 1. Solve the equations :

$$\begin{cases} 3x - 5y = 13 & (1) \\ 2x + 7y = 81 & (2) \end{cases}$$

Note 1. The abbreviations (1), (2), (3), etc., read "equation one," "equation two," etc., are used for convenience to distinguish one equation from another.

Solution. To eliminate x we must make its coefficients equal in both equations. Multiplying the members of (1) by 2, and those of (2) by 3, we have

$$\begin{cases} 6x - 10y = 26 & (3) \\ 6x + 21y = 243 & (4) \end{cases}$$

Subtracting the members of (3) from the corresponding members of (4), we have $31y = 217$. $\therefore y = 7$. Substituting this value of y in (1), we obtain $3x - 35 = 13$. $\therefore x = 16$.

Verification. Substituting 16 for x, and 7 for y in (1) and (2), we have $\begin{cases} 48 - 35 = 13 & (1), \\ 32 + 49 = 81 & (2), \end{cases}$ identities.

Notes: 2. In this solution we eliminate x by subtraction. But suppose we wish to eliminate y instead of x. Multiply (1) by 7, and (2) by 5, then *add* the resulting equations, and we have $31x = 496$. $\therefore x = 16$. This value of x substituted in (1) gives $y = 7$.

3. When one of the unknown numbers has been found, we may use any one of the equations to complete the solution, but it is more convenient to use the one in which the number is less involved.

4. It is usually convenient to eliminate the unknown number which has the smaller coefficients in the two equations. If the coefficients are prime to each other, take each one as the multiplier of the other equation. If they are not prime, find their L. C. M., divide their L. C. M. by the coefficient in each equation, and the quotient will be the smallest multiplier for that equation.

EXAMPLE 2. Solve the equations: $\begin{cases} 15x + 77y = 92 & (1) \\ 55x - 33y = 22 & (2) \end{cases}$

Solution. Multiplying the members of (1) by 11 (the quotient of 165 divided by 15), and those of (2) by 3, we have

$$\begin{cases} 165x + 847y = 1012 & (3) \\ 165x - 99y = 66 & (4) \end{cases}$$

Subtract the members of (4) from the corresponding members of (3), $946y = 946$. $\therefore y = 1$. Substitute this value of y in (1), $15x + 77 = 92$. $\therefore x = 1$.

Proof. Substituting 1 for x, and 1 for y in (1) and (2), we have

$$\begin{cases} 15 + 77 = 92 & (1) \\ 55 - 33 = 22 & (2) \end{cases}$$

Hence, both equations are satisfied for $x = 1$ and $y = 1$.

EXAMPLE 3. Solve the equations: $\begin{cases} 77x - 12y = 289 & (1) \\ 55x + 27y = 491 & (2) \end{cases}$

Process. Multiply (1) by 9, $693x - 108y = 2601$ (3)

Multiply (2) by 4, $220x + 108y = 1964$ (4)

Add (3) and (4), $913x = 4565$. $\therefore x = 5$.

Substitute this value of x in (2), $275 + 27y = 491$. $\therefore y = 8$.

Proof. Substitute 5 for x, and 8 for y in (1) and (2), and we have $\begin{cases} 279 = 279 & (1), \\ 491 = 491 & (2), \end{cases}$ identities.

Let the student supply the method from the solutions.

Exercise 81.

Solve the following simultaneous simple equations:

1. $\begin{cases} 3x + 4y = 10. \\ 4x + y = 9. \end{cases}$

2. $\begin{cases} 8x - y = 34. \\ x + 8y = 53. \end{cases}$

3. $\begin{cases} 10x + 9y = 290. \\ 12x - 11y = 130. \end{cases}$

4. $\begin{cases} 7y - 3x = 139. \\ 2x + 5y = 91. \end{cases}$

5. $\begin{cases} 6x - 5y = -7. \\ 10x + 3y = 11. \end{cases}$

6. $\begin{cases} 9x - 4y = -4. \\ 15x + 8y = -3. \end{cases}$

7. $\begin{cases} 9y + 2x = 15. \\ 4y + 7x = 3. \end{cases}$

8. $\begin{cases} \frac{2}{3}y + \frac{3}{4}x = 26.\text{*} \\ \frac{3}{4}y + \frac{2}{3}x = 25. \end{cases}$

9. $\begin{cases} .25x + 4.5y = 10. \\ .75y - .15x = .9. \end{cases}$

10. $\begin{cases} \dfrac{x}{3} + \dfrac{y}{2} = 7. \\ \dfrac{x}{2} + \dfrac{y}{3} = 8. \end{cases}$

11. $\begin{cases} .5x + 2y = 1.8. \\ .5y - .8x = .08. \end{cases}$

12. $\begin{cases} 7x + \frac{1}{7}y = 99. \\ 7y + \frac{1}{7}x = 51. \end{cases}$

13. $\begin{cases} \frac{7}{2}x + 3y = 22. \\ 11x - \frac{2}{5}y = 20. \end{cases}$

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* Clear of fractions first.

Elimination by Substitution.

90. EXAMPLE. Solve the equations: $\begin{cases} 4x + 3y = 22 & (1) \\ 5x - 7y = 6 & (2) \end{cases}$

Solution. From (2), $x = \dfrac{6 + 7y}{5}$ (3). Since the equations are simultaneous, x means the same thing in both, the substitution of this value of x in (1), will not destroy the equality. Hence, $4\left(\dfrac{6+7y}{5}\right) + 3y = 22$. Clearing of fractions, transposing, and uniting like terms, $43y = 86$. $\therefore\ y = 2$. Substitute this value of y in (3), $x = 4$.

Let the student supply the method.

Exercise 82.

Solve by substitution:

1. $\begin{cases} y + 2x = 13. \\ x + 3y = 14. \end{cases}$

2. $\begin{cases} 7x + 4y = 29. \\ 3x + y = 11. \end{cases}$

3. $\begin{cases} \dfrac{x}{6} + \dfrac{y}{5} = 18. \\ \dfrac{1}{2}y - \dfrac{1}{4}x = 21. \end{cases}$

4. $\begin{cases} .08y - .21x = .33. \\ .7x + .12y = 3.54. \end{cases}$

5. $\begin{cases} \dfrac{x+y}{2} - \dfrac{x-y}{3} = 2\frac{1}{3}. \\ \dfrac{x+y}{3} + \dfrac{x-y}{4} = 11. \end{cases}$

6. $\begin{cases} \frac{1}{4}x + \frac{1}{3}y = 9. \\ \frac{1}{5}x + \frac{1}{4}y = 7. \end{cases}$

7. $\begin{cases} 3y + 4x = 88. \\ 5x + 6y = 61. \end{cases}$

8. $\begin{cases} \dfrac{x}{4} + \dfrac{y}{3} = 1. \\ \dfrac{x}{3} + \dfrac{y}{2} = 1. \end{cases}$

9. $\begin{cases} 3y - 4x = 1. \\ 3x - 2y = 1. \end{cases}$

10. $\begin{cases} \dfrac{4y}{11} - \dfrac{x}{22} = 1. \\ \dfrac{y}{3} - \dfrac{x}{2} = 0. \end{cases}$

11. $\begin{cases} 11y - 7x = 37. \\ 8y + 9x = 41. \end{cases}$ 13. $\begin{cases} 10x = 9 + 7y. \\ 4y = 15x - 7. \end{cases}$

12. $\begin{cases} \dfrac{y}{2} - \dfrac{1}{3}(x-2) - \dfrac{1}{4}(y-3) = 0, \\ y - \dfrac{1}{2}(x-1) - \dfrac{1}{3}(y-2) = 0, \end{cases}$ and verify.

Elimination by Comparison.

91. This method depends upon the following axiom:

6. *Things equal to the same thing are equal to each other.*

EXAMPLE. Solve the equation : $\begin{cases} 6x - 5y = 1 & (1) \\ 7x - 4y = 8\frac{1}{2} & (2) \end{cases}$

Solution. From (1), $x = \dfrac{1+5y}{6}$ (3). From (2), $x = \dfrac{8\frac{1}{2}+4y}{7}$. Since these equations are simultaneous, x means the same thing in both, $\dfrac{1+5y}{6} = \dfrac{8\frac{1}{2}+4y}{7}$. Solving for y, we have $y = 4$. Substituting this value in (3), $x = \dfrac{1+20}{6} = 3\frac{1}{2}$.

Let the student supply the method.

Exercise 83.

Solve by comparison:

1. $\begin{cases} 5x + 6y = -8. \\ 3x + 4y = -5. \end{cases}$ 4. $\begin{cases} 6x + 15y = -6. \\ 8x - 21y = 74. \end{cases}$

2. $\begin{cases} 12y - 7x = 17. \\ 4x + 8y = 20. \end{cases}$ 5. $\begin{cases} \frac{1}{3}x - \frac{1}{5}y = 4. \\ \frac{1}{7}x + \frac{1}{3}y = 8. \end{cases}$

3. $\begin{cases} -5x + 3y = 51. \\ 7x + 2y = 3. \end{cases}$ 6. $\begin{cases} .3y - .7x = .4. \\ .02y + .05x = .22. \end{cases}$

7. $\begin{cases} y = 3x - 19. \\ x = 3y - 23. \end{cases}$

8. $\begin{cases} \frac{3}{4}y + \frac{2}{3}x = 15. \\ \frac{5}{6}y + \frac{7}{9}x = 17. \end{cases}$

9. $\begin{cases} 1.1x - 1.3y = 0. \\ .13x - .11y = .48. \end{cases}$

10. $\begin{cases} .30x - .77y = -2.95. \\ .20x + .21y = 1.65. \end{cases}$

11. $\begin{cases} \dfrac{x+y}{5} - \dfrac{x-y}{2} = 3. \\ \dfrac{x-y}{2} + \dfrac{x+y}{10} = 0. \end{cases}$

12. $\begin{cases} \dfrac{x}{9} + \dfrac{y}{8} = 42. \\ \dfrac{x}{8} + \dfrac{y}{9} = 43. \end{cases}$

92. Each of the equations should be reduced to its simplest form, if necessary, before applying either method of elimination.

Notes: 1. An expeditious method, for the solution of *particular* examples, is that of first adding the given equations, or subtracting one from the other.

2. Usually, in solving examples of two unknown numbers, it is expedient to find the value of the second by substitution; but this is by no means always so.

EXAMPLE. Solve:

$$\begin{cases} y + \dfrac{2y + 4x - 2\frac{12}{19}}{3} = 32 - \dfrac{10\frac{2}{3}y - 5\frac{2}{3}x - 18}{19} & (1) \\ \dfrac{3x+y}{12} + \dfrac{13y - 37\frac{1}{3}}{44} = 12 + \dfrac{9 - 9x - y}{22} - \dfrac{10x + .25y - 10.5}{33} & (2) \end{cases}$$

Process. From (1), $127y + 59x = 1928$ (3)

From (2), $59y + 127x = 1792$ (4)

Adding (3) and (4), $186y + 186x = 3720$ (5)

Dividing (5) by 186, $y + x = 20$ (6)

Subtracting (4) from (3), $68y - 68x = 126$ (7)

Dividing (7) by 68, $y - x = 2$ (8)

Adding (6) and (8), $2y = 22. \quad \therefore y = 11.$

Subtracting (8) from (6), $2x = 18. \quad \therefore x = 9.$

Exercise 84.

Solve:

1. $\begin{cases} y(x+7) = x(y+1). \\ 2y + 20 = 3x + 1. \end{cases}$

2. $\begin{cases} 2y + .4x = 1.2. \\ 3.4y - .02x = .01. \end{cases}$

3. $\begin{cases} (y+1)(x+2) - (y+2)(x+1) = -1. \\ 3(y+3) - 4(x+4) = -8. \end{cases}$

4. $\begin{cases} .3x + .125y = x - 6. \\ 3x - .5y = 28 - .25y. \end{cases}$

5. $\begin{cases} x - 4y = -3. \\ x + y = 32. \end{cases}$

6. $\begin{cases} \dfrac{4x+3y}{10} - \dfrac{2y+7-x}{24} = 5 + \dfrac{x-8}{5}. \\ \dfrac{9x+5y-8}{12} - \dfrac{x+y}{4} = \dfrac{7y+6}{11}. \end{cases}$

7. $\begin{cases} \dfrac{4}{5+y} = \dfrac{5}{12+x}. \\ 2x + 5y = 35. \end{cases}$

8. $\begin{cases} \dfrac{2x+6}{3y+2} = \dfrac{8}{7}. \\ 8x - 4 = 9y. \end{cases}$

9. $\begin{cases} \dfrac{x + \dfrac{y}{2} - 3}{x-5} + 7 = 0. \\ \dfrac{3y - 10(x-1)}{6} + \dfrac{x-y}{4} + 1 = 0. \end{cases}$

10. $\begin{cases} \dfrac{4x-3y-7}{5} = \dfrac{3x}{10} - \dfrac{2y}{15} - \dfrac{5}{6}. \\ \dfrac{y-1}{3} + \dfrac{x}{2} - \dfrac{3y}{20} - 1 = \dfrac{y-x}{15} + \dfrac{x}{6} + \dfrac{1}{10}. \end{cases}$

11. $\begin{cases} \dfrac{x}{2} - 12 = \dfrac{y}{4} + 8. \\ \dfrac{x}{3} - \dfrac{y+x}{5} = \dfrac{3y-2x}{4}. \end{cases}$

12. $\begin{cases} \dfrac{x}{7} + \dfrac{y}{5} = \dfrac{43}{35}. \\ \dfrac{x}{9} + \dfrac{y}{11} = \dfrac{23}{33}. \end{cases}$

13. $\begin{cases} x + \frac{1}{2}(3x - y - 1) = \frac{1}{4} + \frac{3}{4}(y - 1). \\ \frac{1}{5}(4x + 3y) = \frac{7}{10}y + 2. \end{cases}$

14. $\begin{cases} \dfrac{2x}{18} - \dfrac{8y - 2}{36} = 1 - \dfrac{4 + x}{3} + \dfrac{y - x}{6}. \\ 7y = 12x. \end{cases}$

15. $\begin{cases} 2x - 6 = \dfrac{23y - x}{7}. \\ \dfrac{x + 43}{5} + \dfrac{x - y}{6} = 2y. \end{cases}$

16. $\begin{cases} \dfrac{3x}{5} + \dfrac{1 - 3y}{7} = 2\frac{1}{5}. \\ \dfrac{3y + x}{11} - 9 = -y. \end{cases}$

17. $\begin{cases} 2.4x + .32y - \dfrac{.36x - .05}{.5} = .8x + \dfrac{2.6 + .005y}{.25}. \\ \dfrac{.04y + .1}{.3} = \dfrac{.07x - .1}{.6}. \end{cases}$

18. $\begin{cases} y - \dfrac{3x - 2 + y}{11} = 1 + \dfrac{15y + \frac{4}{3}x}{33}. \\ \dfrac{2x + 3y}{6} - \dfrac{x - 5}{4} = \dfrac{11y + 152}{12} - \dfrac{3x + 1}{2}. \end{cases}$

19. $\begin{cases} \dfrac{y - 2}{4} - \dfrac{10 - y}{3} = \dfrac{x - 10}{4}. \\ \dfrac{2x + 4}{3} = \dfrac{x + 4y + 12}{8}. \end{cases}$

20. $\begin{cases} \dfrac{x + 1}{y} = \dfrac{1}{3}. \\ \dfrac{x}{y + 1} = \dfrac{1}{4}. \end{cases}$

21. $\begin{cases} \frac{1}{5}(2x + 7y) - 1 = \frac{2}{3}(2x - 6y + 1). \\ x = 4y. \end{cases}$

22. $\begin{cases} y - \dfrac{2x - y}{23 - y} = 20 - \dfrac{59 - 2y}{2}. \\ x + \dfrac{x - 3}{y - 18} = 30 - \dfrac{73 - 3x}{3}. \end{cases}$

Suggestion. Multiply the members of the first equation by 2, transpose, and unite like terms; then clear the resulting equation of fractions. Multiply the second equation by 3, transpose, and unite like terms; etc.

23. $\begin{cases} \frac{2}{3}x - 4 + \frac{y}{2} + x = 8 - \frac{3y}{4} + \frac{1}{12}. \\ \frac{y}{6} - \frac{x}{2} + 2 = \frac{1}{6} - 2x + 6. \end{cases}$

24. $\begin{cases} \dfrac{6x+9}{4} + \dfrac{3x-5}{4y-6} = 1\tfrac{1}{4} + \dfrac{3x+4}{2}. \\ \dfrac{8x+7}{10} - \dfrac{3x-6y}{2x-8} = 4 - \dfrac{9-4x}{5}. \end{cases}$

25. $\begin{cases} 8x - \dfrac{16+60x}{3y-1} = \dfrac{16xy-107}{5+2y}. \\ 2 + 6y + 9x = \dfrac{27x^2 - 12y^2 + 38}{3x-2y+1}. \end{cases}$

Suggestion. Multiply the members of the first equation by $5+2y$, transpose, and unite like terms; then clear of fractions; etc.

26. $\begin{cases} (\frac{2}{3}x - \frac{5}{12}y) \div \frac{7}{4} - (\frac{3}{2}x - \frac{1}{3}y) \div \frac{23}{2} = 2. \\ \dfrac{x-y}{x+y} = \dfrac{1}{5}. \end{cases}$

27. $\begin{cases} \dfrac{13}{y+2x+3} = -\dfrac{3}{4y-5x+6}. \\ \dfrac{3}{6y-5x+4} = \dfrac{19}{3y+2x+1}. \end{cases}$

28. $\begin{cases} y - x = 1. \\ \dfrac{y+1}{x-1} - \dfrac{y-1}{x} = \dfrac{6}{7}. \end{cases}$

29. $\begin{cases} 5(y+3) = 3(x-2) + 2. \\ \dfrac{2}{y+3} = \dfrac{3}{x-2}. \end{cases}$

30. $\begin{cases} \frac{1}{5}(2y + 7x) - 1 = \frac{2}{3}(2y - 6x + 1). \\ 1\frac{1}{3} = \dfrac{y}{3x}. \end{cases}$

31. $\begin{cases} 6x + 3y + 1 = \dfrac{6y^2 - 24x^2 + 130}{2y - 4x + 3}. \\ 3y - \dfrac{151 - 16y}{4x - 1} = \dfrac{9xy - 110}{3x - 4}. \end{cases}$

32. $\begin{cases} \dfrac{4x + 2y}{16} - \dfrac{4x + 5y}{31} = 0. \\ \dfrac{2x + y}{5} + \dfrac{3y - 2x}{6} = \dfrac{36}{5}. \end{cases}$

33. $\begin{cases} 5x + 20y = .1. \\ 11x + 30y = -.9. \end{cases}$

34. $\begin{cases} \dfrac{x + y}{y - x} - \dfrac{5}{3} = 0. \\ \dfrac{x + y + 1}{y - x - 1} - 7 = 0. \end{cases}$

35. $\begin{cases} x - \dfrac{2x - .5y}{9} + \dfrac{5\frac{1}{2}y - 19x - 15}{171} = 3 + y + \dfrac{y - x + 2}{3}. \\ \dfrac{3x - 10y - 2}{6} - y - 1 = \dfrac{46 + 11y}{57} - .5x. \end{cases}$

93. Fractional simultaneous equations in which the unknown numbers occur in the denominators as *simple* or *like expressions*, are readily solved without previously clearing of fractions. Thus,

EXAMPLE 1. Solve: $\begin{cases} \dfrac{15}{x} + \dfrac{21}{y} = 10 & (1) \\ \dfrac{20}{x} - \dfrac{6}{y} = 2 & (2) \end{cases}$

Solution.

Dividing the members of (2) by 2, we have $\dfrac{10}{x} - \dfrac{3}{y} = 1$ (3). Multiplying the members of (3) by 7, $\dfrac{70}{x} - \dfrac{21}{y} = 7$ (4). Adding the

members of (1) and the corresponding members of (4), we have $\frac{85}{x}=17$. Dividing by 17, $\frac{5}{x}=1$. $\therefore x=5$. Substituting this value of x in (1), gives $\frac{3}{y}=1$. $\therefore y=3$.

Note. If we cleared these equations of fractions they would give the product xy, and thus become quite complex. In the solution of this particular class of examples it is always easier to eliminate one of the unknown numbers without clearing of fractions.

EXAMPLE 2. Solve:

$$\begin{cases} \dfrac{3}{2y}+\dfrac{5}{3x}=34 & (1) \\ \dfrac{2}{3y}+\dfrac{4}{5x}=-8 & (2) \end{cases}$$

Process. Multiply (1) by $\frac{4}{9}$, $\dfrac{2}{3y}+\dfrac{20}{27x}=\dfrac{136}{9}$ (3)

Subtract (3) from (2), $\dfrac{4}{5x}-\dfrac{20}{27x}=-\dfrac{208}{9}$ (4)

Simplify (4), $\dfrac{8}{135x}=-\dfrac{208}{9}$. $\therefore x=-\dfrac{1}{390}$.

Substitute in (2), $\dfrac{2}{3y}-312=-8$, or $\dfrac{2}{3y}=304$. $\therefore y=\dfrac{1}{456}$.

Exercise 85.

Solve:

1. $\begin{cases} \dfrac{2}{x}+\dfrac{1}{y}=10. \\ \dfrac{4}{y}+\dfrac{3}{x}=20. \end{cases}$

2. $\begin{cases} \dfrac{2}{9y}-\dfrac{5}{2x}=-3. \\ \dfrac{5}{3y}+\dfrac{1}{4x}=\dfrac{17}{6}. \end{cases}$

3. $\begin{cases} \dfrac{4}{x}+\dfrac{2}{y}=-5. \\ \dfrac{11}{y}-\dfrac{7}{x}=\dfrac{3}{2}. \end{cases}$

4. $\begin{cases} \dfrac{3}{x}+\dfrac{2}{y}=\dfrac{1}{12}. \\ \dfrac{5}{x}+\dfrac{7}{y}=\dfrac{5}{12}. \end{cases}$

5. $\begin{cases} \dfrac{1}{x} + \dfrac{2}{y} = \dfrac{5}{12}. \\ \dfrac{2}{x} - \dfrac{1}{y} = \dfrac{5}{24}. \end{cases}$

6. $\begin{cases} \dfrac{5}{3y} + \dfrac{2}{5x} = 7. \\ \dfrac{7}{6y} - \dfrac{1}{10x} = 3. \end{cases}$

7. $\begin{cases} \dfrac{3}{x-3} + \dfrac{2}{y+2} = \dfrac{7}{12}. \\ \dfrac{2}{y+2} - \dfrac{3}{x-3} = -\dfrac{1}{12}. \end{cases}$

8. $\begin{cases} \dfrac{1}{x-1} + \dfrac{2}{y+1} = \dfrac{5}{6}. \\ \dfrac{3}{x-1} + \dfrac{4}{y+1} = 2. \end{cases}$

9. $\begin{cases} \dfrac{15}{y} - \dfrac{8}{x} = -\dfrac{17}{3}. \\ \dfrac{2}{y} - \dfrac{3}{x} = -\dfrac{7}{5}. \end{cases}$

10. $\begin{cases} \dfrac{1}{2(x-2)} + \dfrac{4}{3(2y-1)} = 5. \\ \dfrac{3}{5x-10} - \dfrac{5}{4(4y-2)} = 1. \end{cases}$

11. $\begin{cases} \dfrac{5}{x} + \dfrac{7}{2y} = 37\tfrac{1}{2}. \\ \dfrac{3}{2x} - \dfrac{5}{2y} = -6\tfrac{1}{2}. \end{cases}$

12. $\begin{cases} \dfrac{3}{y} - \dfrac{5}{2x} = 16. \\ \dfrac{1}{2y} + \dfrac{4}{x} = -15. \end{cases}$

13. $\begin{cases} \dfrac{5}{x-1} - \dfrac{3}{y-1} = -\dfrac{1}{6}. \\ \dfrac{3}{x-1} - \dfrac{1}{y-1} = \dfrac{1}{30}. \end{cases}$

14. $\begin{cases} \dfrac{5}{2y} + \dfrac{3}{4x} = 7. \\ \dfrac{5}{3y} + \dfrac{3}{2x} = 7\tfrac{1}{3}. \end{cases}$

15. $\begin{cases} \dfrac{5}{x} + \dfrac{16}{y} = 79. \\ \dfrac{16}{x} - \dfrac{1}{y} = 44. \end{cases}$

16. $\begin{cases} \dfrac{11}{2x} + \dfrac{6}{3y} = 17. \\ \dfrac{17}{6y} - \dfrac{5}{x} = -\dfrac{3}{2}. \end{cases}$

17. $\begin{cases} \dfrac{2}{y} - \dfrac{5}{3x} = \dfrac{4}{27}. \\ \dfrac{1}{4y} + \dfrac{1}{x} = \dfrac{11}{72}. \end{cases}$

18. $\begin{cases} \dfrac{2}{x-2} + \dfrac{5}{y+2} = 6. \\ \dfrac{3}{x-2} - \dfrac{1}{y+2} = \dfrac{1}{2}. \end{cases}$ 19. $\begin{cases} \dfrac{1}{2x} - \dfrac{1}{3y} = \dfrac{7}{15}. \\ \dfrac{x - 5y + 4xy}{4xy} = \dfrac{1}{15}. \end{cases}$

19. **Suggestion.** Reduce the first member of the second equation to mixed expressions. Etc.

20. $\begin{cases} \dfrac{2\frac{1}{2}}{3x} - \dfrac{3y}{3y - 2x + 1} = -\dfrac{2}{3}. \\ \dfrac{5}{x} + \dfrac{2y}{3y - 2x + 1} = 6. \end{cases}$ 21. $\begin{cases} \dfrac{12}{x} + \dfrac{25}{y} = 86. \\ \dfrac{25}{x} - \dfrac{16}{y} = 11. \end{cases}$

94. In solving literal simultaneous equations, either of the preceding methods of elimination may be applied, usually the method by addition or subtraction is to be preferred.

Note. Numbers occupying like relations in the same problem, are generally represented by the same letter distinguished by different *subscript* figures; as, a_1; a_2; a_3; etc.; read *a one; a two; a three;* etc.

They may also be represented by different *accents;* as, a'; a''; a'''; etc.; read *a prime; a second; a third;* etc.

EXAMPLE 1. Solve: $\begin{cases} mx + ny = a & (1) \\ m_1x + n_1y = a_1 & (2) \end{cases}$

Process. Multiply (1) by m_1, $\quad m_1mx + m_1ny = m_1a \quad (3)$

Multiply (2) by m, $\quad m_1mx + mn_1y = ma \quad (4)$

Subtract (4) from (3), $\quad m_1ny - mn_1y = m_1a - ma_1,$

or factoring, $\quad (m_1n - mn_1)y = m_1a - ma_1.$

Dividing by $m_1n - mn_1$, $\quad y = \dfrac{am_1 - a_1m}{m_1n - mn_1}.$

Multiply (1) by n_1, $\quad mn_1x + nn_1y = n_1a \quad (5)$

Multiply (2) by n, $\quad m_1nx + nn_1y = na_1 \quad (6)$

Subtract (6) from (5), $\quad mn_1x - m_1nx = n_1a - na_1,$

or factoring, $\quad (mn_1 - m_1n)x = n_1a - na_1.$

Therefore, $\quad x = \dfrac{an_1 - a_1n}{mn_1 - m_1n}.$

EXAMPLE 2. Solve: $\begin{cases} \dfrac{x}{a+b} + \dfrac{y}{a-b} = 2a & (1) \\ \dfrac{x-y}{2ab} = \dfrac{x+y}{a^2+b^2} & (2) \end{cases}$

Process. Free (2) from fractions, transpose, and factor,

$$(a-b)^2x-(a+b)^2y = 0 \quad (3)$$

Simplify (1), $\quad (a-b)x+(a+b)y = 2a(a+b)(a-b) \quad (4)$

Multiply (4) by $a-b$, $\quad (a-b)^2x+(a^2-b^2)y = 2a(a+b)(a-b)^2 \quad (5)$

Subtract (3) from (5), $\quad 2a(a+b)y = 2a(a+b)(a-b)^2.$

Divide by $2a(a+b)$, $\quad y = (a-b)^2.$

Substitute in (1), $\quad \dfrac{x}{a+b} + a - b = 2a. \quad \therefore x = (a+b)^2.$

EXAMPLE 3. Solve: $\begin{cases} \dfrac{m}{n(a+x)} + \dfrac{n-m(m+n)(b-y)}{m(b-y)} = 0 & (1) \\ \dfrac{n}{a+x} + \dfrac{m}{b-y} = m^2+n^2 & (2) \end{cases}$

Process. From (1), $\quad \dfrac{m}{n(a+x)} + \dfrac{n}{m(b-y)} = m+n \quad (3)$

Multiply (3) by $\dfrac{n^2}{m}$, $\quad \dfrac{n}{a+x} + \dfrac{n^3}{m^2(b-y)} = \dfrac{n^2}{m}(m+n) \quad (4)$

Subtract (4) from (2), $\quad \dfrac{m}{b-y} - \dfrac{n^3}{m^2(b-y)} = \dfrac{m^3-n^3}{m}.$

Simplifying, $\quad \dfrac{1}{m(b-y)} = 1. \quad \therefore y = b - \dfrac{1}{m}.$

Substitute $\quad \dfrac{1}{m(b-y)} = 1$ or $\dfrac{m}{b-y} = m^2$ in (2),

$$\frac{n}{a+x} = n^2. \quad \therefore x = \frac{1}{n} - a.$$

EXAMPLE 4. Solve: $\begin{cases} \dfrac{x-y+1}{x-y-1} - a = 0 & (1) \\ \dfrac{x+y+1}{x+y-1} - b = 0 & (2) \end{cases}$

Process. From (1), $(a-1)x-(a-1)y=a+1$ (3)

From (2), $(b-1)x+(b-1)y=b+1$ (4)

Divide (3) by $a-1$, $x-y=\dfrac{a+1}{a-1}$ (5)

Divide (4) by $b-1$, $x+y=\dfrac{b+1}{b-1}$ (6)

Add (5) and (6),
$$2x=\frac{2(ab-1)}{(a-1)(b-1)}.$$
$$\therefore\ x=\frac{ab-1}{(a-1)(b-1)}.$$

Subtract (5) from (6),
$$2y=\frac{2(a-b)}{(a-1)(b-1)}.$$
$$\therefore\ y=\frac{a-b}{(a-1)(b-1)}.$$

Exercise 86.

Solve:

1. $\begin{cases} ax+by=m. \\ bx+ay=n. \end{cases}$

2. $\begin{cases} lx+my=n. \\ px+qy=r. \end{cases}$

3. $\begin{cases} ax=by. \\ bx+ay=c. \end{cases}$

4. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=\dfrac{1}{ab}. \\ \dfrac{x}{a_1}-\dfrac{y}{b_1}=\dfrac{1}{a_1b_1}. \end{cases}$

5. $\begin{cases} \dfrac{3y}{m}+\dfrac{2x}{n}=3. \\ \dfrac{9y}{m}-\dfrac{6x}{n}=3. \end{cases}$

6. $\begin{cases} ax+by=a^2. \\ bx+ay=b^2. \end{cases}$

7. $\begin{cases} px-qy=r. \\ rx-py=q. \end{cases}$

8. $\begin{cases} x+ay=a_1. \\ ax+a_1y=1. \end{cases}$

9. $\begin{cases} \dfrac{y}{m}+\dfrac{x}{n}=a. \\ \dfrac{y}{n}\div\dfrac{x}{m}=-a. \end{cases}$

10. $\begin{cases} \dfrac{y}{a+b}-\dfrac{x}{a-b}=\dfrac{1}{a+b}. \\ \dfrac{y}{a+b}+\dfrac{x}{a-b}=\dfrac{1}{a-b}. \end{cases}$

11. $\begin{cases} a\,y + b\,x = 0. \\ c\,y + b\,x = a. \end{cases}$

13. $\begin{cases} (a-b)\,x = (a+b)\,y. \\ x + y = c. \end{cases}$

12. $\begin{cases} b\,y - r\,n = c\,(m-x). \\ \dfrac{b\,y}{m} + r = c\left(1 + \dfrac{x}{n}\right). \end{cases}$

14. $\begin{cases} x + y = 2. \\ m\,x = n\,y. \end{cases}$

15. $\begin{cases} m\,y = n\,x + \dfrac{m^2 + n^2}{2}. \\ (m-n)\,y = (m+n)\,x. \end{cases}$

16. $\begin{cases} \dfrac{y}{a} + \dfrac{x}{a_1} = 1. \\ \dfrac{y}{a_1} - \dfrac{x}{a} = 1. \end{cases}$

17. $\begin{cases} \dfrac{m}{x} + \dfrac{n}{y} = 1. \\ \dfrac{n}{x} + \dfrac{m}{y} = 1. \end{cases}$

18. $\begin{cases} b\,c\,x = c\,y - 2\,b. \\ b^2\,y + \dfrac{a\,(c^3 - b^3)}{b\,c} = \dfrac{2\,b^3}{c} + c^3\,x. \end{cases}$

19. $\begin{cases} \dfrac{m}{n\,y} + \dfrac{n}{m\,x} = m + n. \\ \dfrac{n}{y} + \dfrac{m}{x} = m^2 + n^2. \end{cases}$

20. $\begin{cases} \dfrac{2}{m\,y} + \dfrac{3}{n\,x} = 5. \\ \dfrac{5}{m\,y} - \dfrac{2}{n\,x} = 3. \end{cases}$

21. $\begin{cases} a\,x - b\,y = a^2 + b^2. \\ (a-b)\,x + (a+b)\,y = 2\,(a^2 - b^2). \end{cases}$

22. $\begin{cases} m\,(m-y) = n\,(x + y - m). \\ m\,(x - n - y) = n\,(x-n). \end{cases}$

23. $\begin{cases} \dfrac{a}{a+x} + \dfrac{b}{b-y} = \dfrac{a}{b}. \\ \dfrac{b}{a+x} - \dfrac{a}{b-y} = \dfrac{b}{a}. \end{cases}$

24. $\begin{cases} \dfrac{x+y+1}{y-x+1} = \dfrac{m+1}{m-1}. \\ \dfrac{x+y+1}{y-x-1} = \dfrac{n+1}{1-n}. \end{cases}$

25. $\begin{cases} y - x + 2\,(m-n) = 0. \\ (x+n)\,(y+m) - (y-m)\,(x-n) = 2\,(m-n)^2. \end{cases}$

95. Simultaneous equations with three or more unknown numbers are solved by eliminating one of the unknown numbers from the given equations; then a second from the resulting equations; and so on, until finally there is but one equation with one unknown number. Thus,

EXAMPLE 1. Solve:
$$\begin{cases} 2y + z + 2v = -23 & (1) \\ y + 3z = -2 & (2) \\ 4x + z = 13 & (3) \\ \dfrac{x}{3} + 3v = -20 & (4) \end{cases}$$

Process. Multiply (2) by 2, $2y + 6z = -4$ (5)

Subtract (5) from (1), $-5z + 2v = -19$ (6)

Multiply (4) by 12, $4x + 36v = -240$ (7)

Subtract (7) from (3), $z - 36v = 253$ (8)

Multiply (8) by 5, $5z - 180v = 1265$ (9)

Add (9) and (6), $-178v = 1246. \quad \therefore v = -7.$

Substitute in (4), $\dfrac{x}{3} - 21 = -20. \quad \therefore x = 3.$

Substitute in (3), $12 + z = 13. \quad \therefore z = 1.$

Substitute in (2), $y + 3 = -2. \quad \therefore y = -5.$

Proof. Substituting -7 for v, 3 for x, -5 for y, and 1 for z in (1), (2), (3), and (4), we have
$$\begin{cases} -23 = -23 & (1), \\ -2 = -2 & (2), \\ 13 = 13 & (3), \\ -20 = -20 & (4), \end{cases}$$
identities.

Note. When the values of several unknown numbers are to be found, it is necessary to have as many simultaneous equations as there are unknown numbers.

EXAMPLE 2. Solve:
$$\begin{cases} \dfrac{1}{2x} + \dfrac{1}{4y} - \dfrac{1}{3z} = \dfrac{1}{4} & (1) \\ \dfrac{1}{x} - \dfrac{1}{3y} = 0 & (2) \\ \dfrac{1}{x} - \dfrac{1}{5y} + \dfrac{4}{z} = 2\frac{2}{15} & (3) \end{cases}$$

Process. Multiply (1) by 2, $\frac{1}{x}+\frac{1}{2y}-\frac{2}{3z}=\frac{1}{2}$ (4)

Subtract (2) from (4), $\frac{5}{6y}-\frac{2}{3z}=\frac{1}{2}$ (5)

Subtract (2) from (3), $\frac{2}{15y}+\frac{4}{z}=2\frac{2}{15}$ (6)

Multiply (5) by 6, $\frac{5}{y}-\frac{4}{z}=3$ (7)

Add (6) and (7), $\frac{77}{15y}=\frac{77}{15}$. $\therefore y=1$.

Substitute in (2), $\frac{1}{x}-\frac{1}{3}=0$. $\therefore x=3$.

Substitute in (5), $\frac{5}{6}-\frac{2}{3z}=\frac{1}{2}$. $\therefore z=2$.

Example 3. Solve:

$$\begin{cases} \frac{1}{x}+\frac{1}{y}=a & (1) \\ \frac{1}{x}+\frac{1}{z}=b & (2) \\ \frac{1}{y}+\frac{1}{z}=c & (3) \end{cases}$$

Process.

Add (1), (2), and (3), $\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=a+b+c$ (4)

Divide (4) by 2, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{a+b+c}{2}$ (5)

Subtract (3) from (5), $\frac{1}{x}=\frac{a+b-c}{2}$. $\therefore x=\frac{2}{a+b-c}$.

Subtract (2) from (5), $\frac{1}{y}=\frac{a+c-b}{2}$. $\therefore y=\frac{2}{a-b+c}$.

Subtract (1) from (5), $\frac{1}{z}=\frac{b+c-a}{2}$. $\therefore z=\frac{2}{b+c-a}$.

Exercise 87.

Solve:

1. $\begin{cases} 2x - y + z = 9. \\ x - 2y + 3z = 14. \\ 3x + 4y - 2z = 7. \end{cases}$

2. $\begin{cases} 4x - 3y + 2z = 40. \\ 5x + 9y - 7z = 47. \\ 9x + 8y - 3z = 97. \end{cases}$

3. $\begin{cases} 2x - 3y + 5z = 15. \\ 3x + 2y - z = 8. \\ -x + 5y + 2z = 21. \end{cases}$

4. $\begin{cases} 3x - 3y + z = 0. \\ 2x - 7y + 4z = 0. \\ 9x + 5y + 3z = 28. \end{cases}$

5. $\begin{cases} x + y + z = 5. \\ 3y - 5x + 7z = 75. \\ 9y - 11z + 10 = 0. \end{cases}$

6. $\begin{cases} .65y - .95x = .5. \\ 5.1x - 3.3z = 6. \\ 20.3z - 23.1x = 21. \end{cases}$

7. $\begin{cases} ax + by + cz = 3. \\ ax - by + cz = 1. \\ ax + by - cz = 1. \end{cases}$

8. $\begin{cases} .2x + .1y + .3z = 14. \\ .5x + .4y + .6z = 32. \\ .7y - .8x + .9z = 18. \end{cases}$

9. $\begin{cases} y + \dfrac{x}{2} + \dfrac{z}{3} = 6. \\ x + \dfrac{z}{2} + \dfrac{y}{3} = -1. \\ z + \dfrac{x}{3} + \dfrac{y}{2} = 17. \end{cases}$

10. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1. \\ \dfrac{x}{a} + \dfrac{y}{c} + \dfrac{z}{b} = 1. \\ \dfrac{x}{b} + \dfrac{y}{a} + \dfrac{z}{c} = 1. \end{cases}$

11. $\begin{cases} \dfrac{1}{x} + \dfrac{2}{y} - \dfrac{3}{z} = 1. \\ \dfrac{5}{x} + \dfrac{4}{y} + \dfrac{6}{z} = 24. \\ \dfrac{7}{x} - \dfrac{8}{y} + \dfrac{9}{z} = 14. \end{cases}$

12. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} - \dfrac{1}{z} = m. \\ \dfrac{1}{x} - \dfrac{1}{y} + \dfrac{1}{z} = n. \\ \dfrac{1}{y} + \dfrac{1}{z} - \dfrac{1}{x} = p. \end{cases}$

13. $\begin{cases} bx + ay = ab. \\ cx + az = ac. \\ cy + bz = bc. \end{cases}$

14. $\begin{cases} v + x + y + z = 14. \\ 2v + x = 2y + z - 2. \\ 3v - x + 2y + 2z = 19. \\ \dfrac{v}{3} + \dfrac{x}{4} + \dfrac{y}{5} + \dfrac{z}{2} = 4. \end{cases}$

15. $\begin{cases} \dfrac{a-x}{x} + \dfrac{b-y}{y} + \dfrac{c-z}{z} = 0. \\ \dfrac{a-x}{x} + \dfrac{b}{y} - \dfrac{c}{z} = 0. \\ \dfrac{2a}{x} - \dfrac{b}{y} - \dfrac{c}{z} = 0. \end{cases}$

15. **Suggestion.** Reduce fractions to mixed expressions. Etc.

16. $\begin{cases} x + 2y = 9. \\ 3y + 4z = 14. \\ 7z + v = 5. \\ 2v + 5x = 8. \end{cases}$

17. $\begin{cases} x + y = 1. \\ y + z = 9. \\ x + z = 5. \end{cases}$

18. $\begin{cases} \dfrac{2}{x} + \dfrac{1}{y} = \dfrac{3}{z}. \\ \dfrac{3}{z} - \dfrac{2}{y} = 2. \\ \dfrac{1}{x} + \dfrac{1}{z} = \dfrac{4}{3}. \end{cases}$

19. $\begin{cases} \dfrac{4y+3x+z}{10} - \dfrac{2x+2z-y+1}{15} = 5 + \dfrac{y-z-5}{5}. \\ \dfrac{9y+5x-2z}{12} - \dfrac{2y+x-3z}{4} = \dfrac{7x+z+3}{11} + \dfrac{1}{6}. \\ \dfrac{5x+3z}{4} - \dfrac{2y+3x-z}{12} + 2z = x - 1 + \dfrac{3y+2x+7}{6}. \end{cases}$

Queries. Upon what principle is elimination by addition and subtraction performed? What substitution? What comparison?

Miscellaneous Exercise 88.

Solve:

1. $\begin{cases} \dfrac{x+1}{y-1} - \dfrac{x-1}{y} = \dfrac{6}{y}. \\ x - y = 1. \end{cases}$

2. $\frac{2}{3}x + \frac{3}{2}y = 16\frac{1}{6} = \frac{3}{2}x - \frac{2}{3}y.$

3. $\begin{cases} a(x+y) + b(x-y) = 1. \\ a(x-y) + b(x+y) = 1. \end{cases}$

4. $\begin{cases} \dfrac{x-a}{b} + \dfrac{y-b}{a} = 0. \\ \dfrac{x+y-b}{a} + \dfrac{x-y-a}{b} = 0. \end{cases}$

5. $\begin{cases} x + 2y = 2 - 3z - 4v. \\ 3y + 2x = 3 - 4z - 5v. \\ 9v - 8z - 3 = -6x - 7y. \\ v = 25 - 4z - 16y - 64x. \end{cases}$

6. $\begin{cases} 4x + y = 11. \\ \dfrac{y}{5x} = \dfrac{7x-y}{3x} - \dfrac{23}{15}. \end{cases}$

7. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = c. \\ \dfrac{x}{b} - \dfrac{y}{a} = 0. \end{cases}$

8. $\begin{cases} \dfrac{y}{3} + \dfrac{x}{2} = 4. \\ \dfrac{y}{2} = \dfrac{x}{3}. \end{cases}$

9. $\begin{cases} \dfrac{y}{a} + \dfrac{x}{b} = 2. \\ \dfrac{y}{a_1} = \dfrac{x}{b_1}. \end{cases}$

10. $\begin{cases} (m^2 - n^2)(5x + 3y) = (4m - n)\,2mn. \\ m^2y - \dfrac{amn^2}{m+n} + (m+n+a)nx = n^2y + (m+2n)mn. \end{cases}$

11. $\begin{cases} 3v + x + 2y - z = 22. \\ 4x - y + 3z = 35. \\ 4v + 3x - 2y = 19. \\ 2v + 4y + 2z = 46. \end{cases}$

12. $\begin{cases} 15x = 24z - 10y + 41. \\ 15y = 12x - 16z + 10. \\ 18y - (7z - 13) = 14x. \end{cases}$

13. $\begin{cases} \dfrac{2}{x} + \dfrac{1}{y} = 3. \\ \dfrac{1}{x} + \dfrac{2}{z} = 11. \\ \dfrac{2}{y} + \dfrac{1}{z} = -3. \end{cases}$

14. $\begin{cases} x + y + z + v = 14. \\ 2x + 3y + 4z + 5v = 54. \\ 4x - 5y - 7z + 9v = 10. \\ 3x + 4y + 2z - 3v = 11. \end{cases}$

16. $\begin{cases} y + z + v = 5. \\ x + z + v = 10. \\ x + y + v = 6. \\ x + y + z = 12. \end{cases}$

15. $\begin{cases} ax + by = 2m. \\ ax + cz = 2n. \\ by + cz = 6p. \end{cases}$

17. $\begin{cases} mx + ny + pz = m. \\ mx - ny - pz = n. \\ mx + py + nz = p. \end{cases}$

18. $\begin{cases} \dfrac{3x - 2y}{3} + 1 + \dfrac{11y - 10}{8} = \dfrac{4x - 3y + 5}{7} + \dfrac{45 - x}{5}. \\ 45 - \dfrac{4x - 2}{3} = \dfrac{55x + 71y + 1}{18}. \end{cases}$

19. $\begin{cases} 7x = 17 + 2z - 3u. \\ v = 23 - 2(z + 2y). \\ u = \frac{5}{2}y - \frac{3}{2}x - 4. \\ y = 2.25 + .75u - .5v. \\ z = 11 - \frac{8}{3}v. \end{cases}$

21. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{m}. \\ \dfrac{1}{x} + \dfrac{1}{z} = \dfrac{1}{p}. \\ \dfrac{1}{z} + \dfrac{1}{y} = \dfrac{1}{n}. \end{cases}$

20. $\begin{cases} ax + bx - cy = m. \\ ay + by - cx = n. \end{cases}$

22. $\begin{cases} x + y + z = a + b + c. \\ a + x = b + y = c + z. \end{cases}$

23. $\dfrac{11 - 7x}{3 - x} + \dfrac{2(5 - 11y)}{11(y - 1)} = \dfrac{17.5 + 5y}{3 - y} - \dfrac{312.5 - 360x}{36(x + 5)} = 5.$

24. $\begin{cases} \dfrac{3}{x} - \dfrac{4}{5y} + \dfrac{1}{z} = 7.6. \\ \dfrac{1}{3x} + \dfrac{1}{2y} + \dfrac{2}{z} = 10.1\dot{6}. \\ \dfrac{4}{5x} - \dfrac{1}{2y} + \dfrac{4}{z} = 16.1. \end{cases}$

25. $\begin{cases} \dfrac{x}{y} + 1 = 4x. \\ \dfrac{x}{z} + 1 = 3x. \\ \dfrac{y}{z} + 1 = 2y. \end{cases}$

26. $$\begin{cases} \dfrac{1}{y+\dfrac{1}{x-\dfrac{5}{y}}} = \dfrac{1}{y-\dfrac{1}{x-\dfrac{7}{y}}}. \\[2ex] \dfrac{1-\dfrac{1}{y}}{x} - 1 = 0. \end{cases}$$

27. $$\begin{cases} x + y = 2\,m^3. \\[1ex] \dfrac{y}{x} = \dfrac{m+n-\dfrac{m\,n}{m+n}}{m-n+\dfrac{m\,n}{m-n}}. \end{cases}$$

28. $$\begin{cases} \dfrac{m}{x} + \dfrac{n}{y} + \dfrac{p}{z} = 3. \\[1ex] \dfrac{m}{x} + \dfrac{n}{y} - \dfrac{p}{z} = 1. \\[1ex] \dfrac{2\,m}{x} - \dfrac{n}{y} - \dfrac{p}{z} = 0. \end{cases}$$

29. $$\begin{cases} a\,x + b\,y + c\,z = 0. \\ a^2\,x + b^2\,y + c^2\,z = 0. \\ a^3\,x + b^3\,y + c^3\,z = 0. \end{cases}$$

30. $$\begin{cases} \dfrac{x\,y}{x+y} = 70. \\[1ex] \dfrac{x\,z}{x+z} = 84. \\[1ex] \dfrac{y\,z}{y+z} = 140. \end{cases}$$

31. $$\begin{cases} \dfrac{x\,y}{x+y} = \dfrac{2}{3}. \\[1ex] \dfrac{y\,z}{y+z} = \dfrac{6}{5}. \\[1ex] \dfrac{x\,z}{x+z} = \dfrac{3}{4}. \end{cases}$$

32. $$\begin{cases} \dfrac{x\,y}{x+y} = m. \\[1ex] \dfrac{x\,z}{x+z} = p. \\[1ex] \dfrac{y\,z}{y+z} = n. \end{cases}$$

33. $$\begin{cases} a\,x + a^2\,y = a^2\,y + a^3\,z = 2. \\ a^3\,z + a^4\,x = a^3 + 1. \end{cases}$$

34. $$\frac{x-2z}{3\,x-2\,y} = \frac{2\,(3\,x-2\,y)}{3\,z-7} = 1 = \frac{5\,z-y}{2\,x-3\,z}.$$

35. $$\begin{cases} \dfrac{y}{x} = \dfrac{m-n+\dfrac{n^2}{m-n}\left(1-\dfrac{n\,(m+n)}{m^2+m\,n+n^2}\right)}{\dfrac{m^2}{m+n}+\dfrac{n^3}{m^2+m\,n+n^2}}. \\[2ex] y - x = 2\,n^5. \end{cases}$$

CHAPTER XVII.

PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS.

96. THE solutions of the following problems lead to simultaneous simple equations of two or more unknown numbers. In the solution of such problems the conditions must be sufficient to give just as *many equations as there are unknown numbers to be determined.*

Exercise 89.

1. If 5 be added to both numerator and denominator of a fraction, its value is $\frac{3}{4}$; and if 3 be subtracted from both numerator and denominator, its value is $\frac{1}{2}$. Find the fraction.

Suggestion. Let $x =$ the numerator,
and $y =$ the denominator.

By the conditions, $\begin{cases} \dfrac{x+5}{y+5} = \dfrac{3}{4}, \\[2mm] \dfrac{x-3}{y-3} = \dfrac{1}{2}. \end{cases}$

Solving these equations, $x = 7$, $y = 11$.
Therefore, the fraction is $\frac{7}{11}$.

2. A certain fraction becomes equal to 3 when 9 is added to its numerator, and equal to 2 when 2 is subtracted from its denominator. Find the fraction.

3. Find two fractions with numerators 5 and 3, respectively, whose sum is $\frac{29}{24}$, and if their denominators are interchanged their sum is $\frac{9}{8}$.

4. A certain fraction becomes equal to $\frac{2}{3}$ when the denominator is increased by 3, and equal to $\frac{1}{3}$ when the numerator is diminished by 3. Find the fraction.

5. A fraction which is equal to $\frac{3}{5}$ is increased to $\frac{13}{19}$ when a certain number is added to both its numerator and denominator, and is $\frac{1}{4}$ when 3 more than the same number is subtracted from each. Find the fraction.

6. If a be added to the numerator of a certain fraction, its value is a; and if a be added to its denominator, its value is $\frac{1}{2}(a-1)$. Find the fraction.

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7. Find two numbers, such that two times the greater added to one fifth the less is 36; three times the greater subtracted from eight times the less, and the remainder divided by 9, the quotient is $7\frac{5}{6}$.

8. Find two numbers, such that if the first be increased by a, it will be m times the second, and if the second be increased by b, it will be n times the first.

9. Find two numbers, such that if to $\frac{1}{3}$ of the sum you add 18, the result will be 21; and if from $\frac{2}{5}$ their difference you subtract $\frac{3}{4}$, the remainder is 3.65.

10. A farmer sold to one person 25 bushels of corn and 52 bushels of oats for \$33.30; to another person 42 bushels of corn, and 37 bushels of oats for \$35.80. Find the number of dollars per bushel received for each.

11. A farmer sold a bushels of corn and b bushels of oats for m dollars; also at the same time, c bushels of corn and d bushels of oats for n dollars. Find the number of dollars per bushel received. Apply the result to 10.

12. A grocer bought a certain number of eggs, part at 2 for 3 cents and the rest at 5 for 8 cents, paying $7.50 for the whole. He sold them at $23\frac{3}{4}$ cents a dozen, and made $2 by the transaction. How many of each kind did he buy?

13. A grocer bought a certain number of eggs, part at the rate of a eggs for m cents and the rest at the rate of b eggs for n cents, and paid c dollars for the whole. He sold them at d cents a dozen, and made p dollars by the transaction. How many of each kind did he buy? Apply the result to 12.

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14. A number is expressed by three digits. The sum of the digits is 8; the sum of the first and second exceeds the third by 4; and if 99 be added to the number, the digit in the units' and hundreds' place will be interchanged. Find the numbers.

Suggestion. Let $z =$ the digit in units' place,
and $y =$ the digit in tens' place,
also $x =$ the digit in hundreds' place.

Hence, $100x + 10y + z =$ the number,
and $100z + 10y + x =$ the number with the digit in units' and hundreds' place interchanged.

By the conditions,

$$\begin{cases} x + y + z = 8, \\ x + y - 4 = z, \\ 100x + 10y + z + 99 = 100z + 10y + x. \end{cases}$$

Solving these equations, $z = 2,\ y = 5,\ x = 1.$

Therefore, the number is 152.

Note 1. In verifying, the results should be tested directly by the *conditions of the problem*. Thus, in the above, the sum of 2, 5, and 1 is, as one condition requires, 8. The sum of 1 and 5 exceeds 2 by 4. The sum of 152 and 99 is 251 as required.

15. A number is expressed by three digits. The middle digit is twice the left hand digit, and one less than the right hand digit. If 297 be added to the number, the order of the digits will be reversed. Find the number.

16. A number is expressed by three digits. The sum of the digits is 18; the number is equal to 99 times the sum of the first and third digits, and if 693 be subtracted from the number, the digit in the units' and hundreds' place will be interchanged. Find the number.

17. The sum of the three digits of a number is n; the number is equal to a times the sum of the first and third digits, and if m be subtracted from the number, the digit in the units' and hundreds' place will be interchanged. Find the number.

18. If a certain number be divided by the sum of its two digits the quotient is 3, and the remainder 3; if the digits be interchanged, and the resulting number be divided by the sum of the digits, the quotient is 7, and the remainder 9. Find the number.

19. If a certain number be divided by the sum of its two digits the quotient is a, and the remainder b; if the digits be interchanged, and the resulting number be divided by the sum of the digits, the quotient is c, and the remainder m. Find the number.

20. The sum of the three digits of a number is 16. If the number be divided by the sum of its hundreds' and units' digits the quotient is 77 and the remainder 6; and if it be divided by the number expressed by its two right-hand digits, the quotient is 16 and the remainder 5. Find the number.

21. The sum of the three digits of a number is 9. If the number be divided by the difference of its hundreds' and units' digits, the quotient is 157, and the remainder 1; and if it be divided by the number expressed by its two right-hand digits, the quotient is 21. Find the number.

22. A, B, and C can together do a piece of work in 12 days; A and B can together do it in 20 days; B and C can together do it in 15 days. Find the time in which each can do the work.

Suggestion Let $x =$ the *number* of days in which A can do it,
and $y =$ the *number* of days in which B can do it,
also $z =$ the *number* of days in which C can do it.

The equations are $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{12}$, $\frac{1}{x}+\frac{1}{y}=\frac{1}{20}$, and $\frac{1}{y}+\frac{1}{z}=\frac{1}{15}$; from which $x=60$ and $y=z=30$.

23. A and B can do a piece of work together in 48 days; A and C in 30 days; B and C in $26\frac{2}{3}$ days. How many days will it take each, and how many altogether, to do it?

24. A and B can do a piece of work together in a days; but if A had worked m times as fast, and B n times as fast, they would have finished it in c days. How many days will it take each to do it?

25. A drawer will hold 24 arithmetics and 20 algebras; 6 arithmetics and 14 algebras will fill half of it. How many of each will it hold?

26. A purse holds 19 crowns and 6 guineas; 4 crowns and 5 guineas fill $\frac{17}{63}$ of it. How many will it hold of each?

27. A purse holds c crowns and a guineas; c_1 crowns and a_1 guineas will fill $\frac{m}{n}$th of it. How many will it hold of each?

28. A and B together could have completed a piece of work in 15 days, but after laboring together 6 days, A was left to finish it alone, which he did in 30 days. In how many days could each have performed the work alone?

29. Two persons, A and B, could finish a piece of work in m days; they worked together a days when B was called off and A finished it in n days. In how many days could each do it?

30. A can row 8 miles in 40 minutes down stream, and 14 miles in 1 hour and 45 minutes against the stream. Find the number of miles per hour that the stream flows, also that A rows in still water.

Suggestion. Let $x =$ the *number* of miles per hour that A can row in still water,

and $y =$ the *number* of miles per hour that the stream flows.

Then, $x + y =$ the *number* of miles per hour that A can row down the stream,

and $x - y =$ the *number* of miles per hour that he can row up the stream.

Since the distance divided by the rate will give the time, by the conditions,

$$\begin{cases} \dfrac{8}{x+y} = \dfrac{2}{3}, \\ \dfrac{14}{x-y} = 1\tfrac{3}{4}. \end{cases}$$

31. A can row m miles in h hours down stream, and m_1 miles in h_1 hours against the stream. Find the number of miles per hour that the stream flows, also that A rows in still water. Apply the result to problem 30.

32. A boatman sculls down a stream, which runs at the rate of 5 miles an hour, for a certain distance in 3 hours, and finds that it takes him 13 hours to return. Find the distance sculled down stream, and his rate of rowing in still water.

33. A man who can row at the rate of 15 miles an hour down stream, finds that it takes 3 times as long to come up the stream as to go down. Find the number of miles per hour that the stream flows.

34. A waterman rows 30 miles and back in 12 hours; and he finds that he can row 3 miles against the stream in the same time as 5 miles with it. Find the number of hours in going and coming respectively; also, the number of miles per hour of the stream.

35. A waterman can row down stream a distance of m miles and back again in h hours; and he finds that he can row b miles against the stream in the same time he rows a miles with it. Find the number of hours in going and coming, respectively; also the number of miles per hour of the stream, and his rate of rowing in still water.

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36. Five pounds of sugar and 3 pounds of tea cost \$2.05, but if the price of sugar was to rise 40%, and the price of tea 20% they would cost \$2.82. Find the number of cents in the cost of a pound of each.

37. If l pounds of sugar and l_1 pounds of tea cost m dollars, and the price of sugar was to rise a%, and the price of tea b%, they would cost n dollars. Find the number of cents in the cost of a pound of each.

38. The amount of a sum of money, at simple interest, for 11 months is \$1055; and for 17 months it is \$1085. Find the sum and the rate per cent of interest.

39. The amount of a sum of money, at simple interest, for m months is a dollars; and for n months it is b dollars. Find the sum and the rate of interest.

40. A grocer mixes three kinds of coffee. He can sell a mixture containing 2 pounds of the first kind, 9 pounds of the second, and 5 pounds of the third, at 18 cents per pound; or one composed of 6 pounds of the first, 6 pounds of the second, and 9 pounds of the third, at 19 cents per pound; or one composed of 5 pounds of the first kind, 2 pounds of the second, and 18 pounds of the third, at 22 cents per pound. Find the number of cents in the cost of a pound of each kind.

41. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; if the circumference of the fore-wheel be increased by $\frac{1}{4}$ of its present size, and the circumference of the hind-wheel by $\frac{1}{5}$ of its present size, the 6 will be changed to 4. Find the number of yards in the circumference of each wheel.

42. The fore-wheel of a carriage makes a revolutions more than the hind-wheel in going b feet. If the circumference of the fore-wheel be increased by $\frac{m}{n}$th of itself, and that of the hind-wheel by $\frac{s}{r}$th of itself, the hind-wheel will make c revolutions more than the fore-wheel. Find the circumference of each wheel.

43. A grocer has two kinds of coffee. He sells a pounds of the first kind, and b pounds of the second, for m dollars; or, a_1 pounds of the first kind, and b_1 pounds of the second, for m_1 dollars. Find the number of dollars in the price of a pound of each kind.

44. A jeweller has two silver cups, and for the two a single cover worth 90 cents. If he puts the cover upon the first cup it will be worth $1\frac{1}{2}$ times as much as the other; if he puts it upon the second cup it will be worth $1\frac{1}{12}$ times as much as the first. How many dollars in the value of each cup?

45. A jeweller has two silver cups, and for the two a single cover worth a dollars. If he puts the cover upon the first cup, it will be worth m times as much as the other; if he puts it upon the second cup it will be worth n times as much as the first. How many dollars in the value of each cup?

46. A broker invests \$5000 in 3's, \$4000 in 4's, and has an income from both investments of \$315.50. If his investment had been \$1000 more in the 3's, and less in the 4's, his income would have been \$5.50 greater. Find the market value of each class of bonds.

Note 2. 3's means bonds which bear 3% interest. The "quoted" price of a bond is its *market value*. Thus, a bond quoted at $115\frac{1}{2}$ means that a \$100 bond can be bought for \$115.50 *in the market.*

47. A broker invests m dollars in a's, n dollars in c's, and has an income from both investments of b dollars. If his investment had been d dollars less in the a's, and more in the c's, his income would have been p dollars less. Find the price paid for each kind of bonds.

48. A and B do a piece of work together in 30 days, for which they are to receive \$160. But A is idle 8 days and B is idle 4 days, in consequence of which the work occupies $5\frac{1}{2}$ days more than it would otherwise have done. Find the number of dollars received by each.

49. A and B do a piece of work together in m days, for which they are to receive c dollars. But A is idle a days and B is idle b days, in consequence of which the work occupies n days more than it would otherwise have done. Find the number of dollars received by each.

50. The amount of a sum of money, at simple interest, for 5 years is \$600; and for 8 years it is \$660. Find the number of dollars in the sum, and the rate of interest.

51. The amount of a sum of money, at simple interest, for a years is m dollars; and for b years it is n dollars. Find the number of dollars in the sum, and the rate of interest.

52. If a grocer sells a box of tea at 30 cts. a pound, he will make \$1, but if he sells it at 22 cts. a pound, he will lose \$3. Find the number of pounds in the box, and the number of cents in the cost of a pound.

53. The smaller of two numbers divided by the larger is .21, with a remainder .04162. The greater divided by the smaller is 4, with .742 for a remainder. Find the numbers.

54. The smaller of two numbers divided by the larger is a, with a remainder m. The greater divided by the smaller is b, with c for a remainder. Find the numbers.

CHAPTER XVIII.

EXPONENTS.

97. AN **Exponent** is a figure or term written at the right of and above a number or term (Art. 21).

Thus, in the expressions 5^2, a^c, $b^{\frac{m}{n}}$, and $(a+b)^3$; 2, c, $\frac{m}{n}$, and 3 are exponents.

Zero Exponents. When the dividend and divisor are equal the quotient is 1.

Thus, $\frac{3^2}{3^2}=1;\quad \frac{a^4}{a^4}=1;\ \frac{a^6}{a^6}=1;\ \frac{a^n}{a^n}=1;$ etc.

But (Art. 30), $\frac{3^2}{3^2}=3^{2-2}=3^0;\ \frac{a^4}{a^4}=a^0;\ \frac{a^6}{a^6}=a^0;\ \frac{a^n}{a^n}=a^0;$ etc.

Therefore, it follows that $a^0=1$. Hence, in general,

I. *Any expression with zero for an exponent is* 1.

The **Reciprocal** of a number is *unity* divided by that number.

Thus, the reciprocal of n is $\frac{1}{n}$; of $n+m$ is $\frac{1}{n+m}\cdot$

Negative Integral Exponents.

$$a^3 \times a^{-3} = a^{3-3} = a^0 = 1.$$

Divide by a^3, $\qquad a^{-3}=\frac{1}{a^3}\cdot$

$$a^n \times a^{-n} = a^{n-n} = a^0 = 1.$$

Divide by a^n, $\qquad a^{-n}=\frac{1}{a^n}\cdot$ Hence, in general,

II. *A negative integral exponent indicates the reciprocal of the expression with a corresponding positive exponent.*

The expression a^n, where n is any positive integer, represents *the product of* n *equal factors, each equal to* a. It has been shown that:

Art. 21, $a^m \times a^n = a^{m+n}$.

Art. 30, $a^m \div a^n = a^{m-n}$, where m is *greater* than n.

Art. 30, $a^m \div a^n = \dfrac{1}{a^{n-m}}$, where m is *less* than n.

Art. 27, $(a^m)^n = a^{mn}$, whatever the value of m.

Thus,

By Art. 21, $a^b \times a^c \times a^d \times \dots . a^m = a^{b+c+d+\dots . m}$.

Take n factors of a^b, a^c, a^d, ... a^m, and suppose each of the n exponents equal to m, then it follows that

$(a^m)^n = a^{mn}$. Hence, m can be *positive* or *negative*, *integral* or *fractional*.

By II., $a^{-n} = \dfrac{1}{a^n}$.

Multiply by a^m, $a^{-n} \times a^m = \dfrac{a^m}{a^n}$.

If m is *greater* than n, Art. 30, $\dfrac{a^m}{a^n} = a^{m-n}$.

Therefore, $a^{-n} \times a^m = a^{m-n}$.

If m is *less* than n, $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$.

By II., $\dfrac{1}{a^{n-m}} = a^{m-n}$.

Therefore, $a^{-n} \times a^m = a^{m-n}$ for all possible integral values of m and n.

98. By Art. 27, $(ab)^n = a^n \times b^n$.

Therefore, $a^n b^n = (ab)^n$.

Similarly, $a^n \times b^n \times c^n \times \dots . p^n = (abc \dots . p)^n$.

If n is a negative integer,

$$a^{-n} \times b^{-n} = \frac{1}{a^n \times b^n} = \frac{1}{(a\,b)^n} = (a\,b)^{-n}.$$

Similarly,

$a^{-n} \times b^{-n} \times c^{-n} \times \ldots . p^{-n} = (a\,b\,c\, \ldots .\, p)^{-n}$. Hence, in general,

I. *The product of two or more factors, each affected with the same exponent, is the same as their product affected with the exponent.*

By II., Art. 97, $\qquad a^n \div b^n = a^n b^{-n}.$

Also, $\qquad a^n b^{-n} = (a\,b^{-1})^n = \left(\frac{a}{b}\right)^n.$

Therefore, $\qquad a^n \div b^n = \left(\frac{a}{b}\right)^n.$

Similarly, $\qquad a^{-n} \div b^{-n} = \left(\frac{a}{b}\right)^{-n}.$ Hence, in general,

II. *The quotient of any two factors, each affected with the same exponent, is the same as their quotient affected with the exponent.*

Illustrations: I. $2^2 \times 3^2 = (2 \times 3)^2 = (6)^2 = 36$; $2^3 \times 3^3 \times 4^3 = (2 \times 3 \times 4)^3 = (24)^3 = 13824$; $2^{-2n} \times 3^{-2n} \times 4^{-2n} = (2 \times 3 \times 4)^{-2n} = [(24)^2]^{-n} = (576)^{-n} = \frac{1}{576^n}$; $(\frac{2}{3})^{-2} \times (\frac{3}{4})^{-2} \times (\frac{1}{2})^{-2} = (\frac{2}{3} \times \frac{3}{4} \times \frac{1}{2})^{-2} = (\frac{1}{4})^{-2} = \frac{1}{(\frac{1}{4})^2} = 16.$

II. $24^2 \div 6^2 = (\frac{24}{6})^2 = (4)^2 = 16$; $(-16)^{-3} \div (-4)^{-3} = \left(\frac{-16}{-4}\right)^{-3} = (4)^{-3} = \frac{1}{64}$; $(a^{-1})^{4m} \div (a)^{4m} = \left(\frac{a^{-1}}{a}\right)^{4m} = \left(\frac{1}{a^2}\right)^{4m} = \frac{1}{a^{8m}}.$

These examples are said to be simplified, that is, they are expressed in their simplest forms.

Exercise 90.

Simplify:

1. $(n^2)^2 \times (n^3)^2 \times (n)^2$; $(\frac{1}{2})^2 \times (2)^2 \times (\frac{2}{3})^2 \div (\frac{4}{3})^2$.

2. $(x^n y^{-m})^3 \div (x^{-n} y^{2m})^3$; $(216\, x^2)^4 \div (54\, y^{-2})^4$.

3. $(\frac{4}{3}\, a)^3 \times (\frac{9}{2}\, x^{-2})^3$; $\left(x^{\frac{1}{n}}\right)^{-n} \times \left(x^{\frac{2}{x}}\right)^{-n}$; $(x)^4 \times (x^{-\frac{1}{4}})^4$.

4. $(x)^n \times \left(x^{\frac{1}{n}}\right)^n$; $(\frac{2}{3})^{-n} \times (\frac{3}{4})^{-n} \times (2)^{-n}$; $(a^{-2}\, b)^5 \times (a\, b^{-3})^5$.

5. $(2\, n)^{10} \times (2^{-1}\, m)^{10}$; $(a\, b^{-1}\, c^{-2})^3 \div (a^{-1}\, b^{-2}\, c^{-4}\, m^n)^3$.

6. $(4\, a^{4b}\, x^6)^{-n} \div (2^{-2}\, a^{-3b}\, x^{-5}\, y^n)^{-n}$; $(x^{-1}\, y^{\frac{1}{2}})^{-3} \div (x^{\frac{1}{2}}\, y^{-1})^{-3}$.

7. $a^{-\frac{m}{n}} \times \left(3\, b^{\frac{1}{n}}\right)^{-m} \times \left(c^{\frac{1}{n}}\right)^{-m}$; $(x)^{1-a} \div \left(x^{\frac{1}{a}}\right)^{1-a}$.

8. $(a^{-2}\, b)^{-2} \times (a\, b^{-3})^{-2}$; $(a^3\, b^3 + a^6)^{-3} \div (a^6 - a^3\, b^3)^{-3}$.

9. $(\frac{1}{5})^n \times (\frac{10}{3})^n \times (\frac{3}{4})^n$; $(a^{2n} + a^n\, b^{2n})^{-1} \times (a^n - b^{2n})^{-1}$.

10. $(a^{-\frac{1}{2}})^{-5} \times (x^{\frac{1}{3}})^{-5} \times (x^{-\frac{2}{3}})^{-5} \times (a^{\frac{3}{2}})^{-5} \times (b^{\frac{1}{5}})^{-5}$.

11. $(a^{n+k})^n \times (b^{2n-k})^n \times \left(a^{\frac{1}{n}}\right)^n \times \left(b^{-\frac{1}{n}}\right)^n$; $a^3 \div (2\, a)^3$.

12. $a^{-2} \times (2\, a)^{-2} \div \left(\dfrac{a}{2}\right)^{-2}$; $(2\, a^{-2})^{-2} \times \left(\dfrac{a}{2}\right)^{-2} \times (\frac{3}{4}\, a)^{-2}$.

13. $(a^{-\frac{1}{2}}\, \sqrt[3]{x})^{-3} \times (x^{-2}\sqrt{a^{-6}})^{-3}$; $\left(a^{\frac{1}{n}}\right)^{2n} \times \left(b^{\frac{1}{n}}\right)^{2n} \times (c^n)^{2n}$.

14. $(2^n)^{-n} \times (2^{n-1})^{-n} \times (2^{-2n-1})^{-n} \times (2^{-2n+1})^{-n} \times (4^n)^{-n}$.

15. $(2^{n+1})^m \times (2^{-n^2+n})^m \times (2^{n^2-1})^m \times (4^{-n-1})^m \div (16)^{-m}$.

16. $[(x-y)^{-3}]^n \times [(x+y)^n]^{-3}$; $(\frac{2}{5})^{-n} \times (\frac{5}{3})^{-n} \div (\frac{4}{3})^{-n}$.

17. $\left(\dfrac{a^{-2}\, b}{a^3 b^{-4}}\right)^{-2} \div \left(\dfrac{a\, b^{-1}}{a^{-3} b^2}\right)^{-2}$; $\left(\dfrac{a^{\frac{1}{3}}}{b^{-\frac{1}{4}}}\right)^2 \times \left(\dfrac{b^{\frac{1}{4}}}{a^{\frac{1}{3}}}\right)^2 \times \left(\dfrac{a^{-\frac{1}{3}}}{b^{-\frac{1}{2}}}\right)^2$.

99. **Positive Fractional Exponents.** If m and n are both positive integers,

$$\left(a^{\frac{m}{n}}\right)^n = a^m.$$

Take the nth root of both members, $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

Therefore, $a^{\frac{m}{n}}$ means the nth root of the mth power of a, or the mth power of the nth root of a. Hence,

The numerator, in a fractional exponent, denotes a power, and the denominator a root.

The denominator of the exponent corresponds to the *index* of the root. Thus, $(81)^{\frac{3}{4}} = \sqrt[4]{(81)^3} = (\sqrt[4]{81})^3 = (3)^3 = 27$.

In $a^{\frac{m}{n}} = \sqrt[n]{a^m}$, m is the *index* of the power, and n is the *index* of the root; also a, m, and n may be any numbers. The expression may be raised to the power indicated by the numerator of the exponent and then extract the root of the result indicated by the denominator; or, extract the root first and then raise the result to the power indicated by the numerator of the exponent. Thus,

$$(-8)^{\frac{2}{3}} = \sqrt[3]{(-8)^2} = \sqrt[3]{64} = 4; \text{ or, } (-8)^{\frac{2}{3}} = (\sqrt[3]{-8})^2 = (-2)^2 = 4.$$

Notes: 1. a^{-n} is read "a exponent $-n$;" $a^{\frac{m}{n}}$ is read "a exponent $\frac{m}{n}$;" $a^{-\frac{m}{n}}$ is read "a exponent $-\frac{m}{n}$." These are abbreviated forms for "a with an exponent $-n$;" etc.

2. It is manifestly incorrect to read $a^{\frac{m}{n}}$ "the $\frac{m}{n}$th power of a." There is no such thing as a *fractional* power.

3. We must be careful to notice the difference between the signification of a fraction used as an exponent, and its common signification. Thus, $\frac{4}{5}$ used as an exponent signifies that a number is resolved into five equal factors, and the product of four of them taken.

100. By Art. 73, $a^{\frac{m}{n}} = a^{\frac{m \times c}{n \times c}} = a^{\frac{mc}{nc}}$;

also, $a^{\frac{m}{n}} = a^{\frac{m \div c}{n \div c}} = a^{\frac{\frac{m}{c}}{\frac{n}{c}}}$. Hence,

I. *Multiplying or dividing the terms of a fractional exponent by the same number will not change the value of the expression.*

$$a^{\frac{1}{mn}} = a^{\frac{1}{m} \times \frac{1}{n}}.$$

But $$a^{\frac{1}{m} \times \frac{1}{n}} = \sqrt[n]{a^{\frac{1}{m}}},$$

and $$\sqrt[n]{a^{\frac{1}{m}}} = \sqrt[m]{\sqrt[n]{a}}.$$

Therefore, $$a^{\frac{1}{mn}} = \sqrt[m]{\sqrt[n]{a}}.$$ Hence, in general,

II. *The* mn*th root of a number is equal to the* m*th root of the* n*th root of that number.*

Illustrations.

$$2^{\frac{1}{3}} = 2^{\frac{4}{12}}; \; b^{\frac{6}{9}} = b^{\frac{2}{3}}; \; b^{2c} = b^{\frac{6c^2}{3c}}; \; \sqrt[6]{64} = \sqrt[3]{\sqrt{64}} = \sqrt[3]{8} = 2.$$

101. Negative Fractional Exponents. If m and n are both positive integers,

$$\left(a^{-\frac{m}{n}}\right)^n = a^{-m}.$$

By II., Art. 97, $$a^{-m} = \frac{1}{a^m}.$$

Take the nth root of both members,

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}.$$ Hence,

Any expression affected with a negative fractional exponent is equal to the reciprocal of the expression with a corresponding positive exponent.

Notes: 1. From the relation $a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}$, $a^{\frac{m}{n}} = \frac{1}{a^{-\frac{m}{n}}}$. Hence, the method of Art. 30 is true for fractional exponents.

2. Any factor of the dividend may be removed to the divisor (or from the numerator to the denominator of a fraction), or any factor of the divisor to the dividend, by changing the sign of its exponent.

Illustrations. $2^{-\frac{2}{3}} = \frac{1}{2^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{4}}$; $(\frac{9}{4})^{-\frac{1}{2}} = \frac{1}{(\frac{9}{4})^{\frac{1}{2}}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$; $\frac{1}{a^{\frac{2}{3}}} = a^{-\frac{2}{3}}$;
$[(\frac{1}{2})^{-\frac{2}{3}} \times (\frac{1}{3})^{-\frac{1}{6}} \times 4^{-\frac{1}{2}}]^6 = (\frac{1}{2})^{-\frac{2}{3}\times 6} \times (\frac{1}{3})^{-\frac{1}{6}\times 6} \times 4^{-\frac{1}{2}\times 6} = (\frac{1}{2})^{-4}$
$\times (\frac{1}{3})^{-1} \times 4^{-3} = \frac{1}{(\frac{1}{2})^4} \times \frac{1}{\frac{1}{3}} \times \frac{1}{4^3} = \frac{2^4 \times 3}{2^6} = \frac{3}{4}$; $\left(x^{-1} \div x^{-\frac{1}{n}}\right)^n$
$= x^{-n} \div x^{-1} = \frac{1}{x^n} \div \frac{1}{x} = \frac{x}{x^n} = x^{1-n}$.

102.
$$\left(a^{\frac{1}{n}} b^{\frac{1}{n}}\right)^n = a\,b.$$
Take the nth root of both members,
$$a^{\frac{1}{n}} b^{\frac{1}{n}} = (a\,b)^{\frac{1}{n}}.$$
Similarly,* $a^{\frac{1}{n}} \times b^{\frac{1}{n}} \times c^{\frac{1}{n}} \times \ldots\ldots p^{\frac{1}{n}} = (a\,b\,c\ldots\ldots p)^{\frac{1}{n}}$. Hence,

The product of two or more factors each affected with the same root index, is the same as their product affected with the root index.

In the same manner we can prove that
$$a^n \div b^{\frac{1}{n}} = \left(\frac{a}{b}\right)^{\frac{1}{n}}.$$

Notes: * 1. If we suppose that there are m factors of a, b, c, $\ldots\ldots p$, and that each factor is equal to a, then it follows that
$$\left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}}.$$
By Art. 99, $\left(a^m\right)^{\frac{1}{n}} = a^{\frac{m}{n}}$.

Therefore, $\left(a^{\frac{1}{n}}\right)^m = a^{\frac{m}{n}}$.

2. Similarly, $\left(a^{\frac{1}{n}}\right)^n = a$. Hence,

The nth power of the nth root of a number is equal to that number.

Illustrations. $(\frac{4}{9})^{\frac{1}{3}} \times (\frac{2}{3})^{\frac{1}{3}} \times 8^{\frac{1}{3}} = (\frac{4}{9} \times \frac{2}{3} \quad 8)^{\frac{1}{3}} = \sqrt[3]{\frac{64}{27}} = \frac{4}{3}$;
$(a^m)^{\frac{1}{c}} \times (b^n)^{\frac{1}{c}} \times (p^x)^{\frac{1}{c}} = (a^m b^n p^x)^{\frac{1}{c}} = \sqrt[c]{a^m b^n p^x}$; $(\frac{2}{27})^{\frac{1}{4}} \div (\frac{3}{8})^{\frac{1}{4}}$
$= (\frac{2}{27} \div \frac{3}{8})^{\frac{1}{4}} = \sqrt[4]{\frac{16}{81}} = \frac{2}{3}$; $(a^m)^{\frac{1}{c}} \div (b^n)^{\frac{1}{c}} = \left(\frac{a^m}{b^n}\right)^{\frac{1}{c}} = \sqrt[c]{\frac{a^m}{b^n}}$.

103. $$\left(a^{\frac{m}{n}} \times a^{\frac{b}{c}}\right)^{nc} = a^{mc+nb}.$$

Take the ncth root, $$a^{\frac{m}{n}} \times a^{\frac{b}{c}} = (a^{mc+nb})^{\frac{1}{nc}}.$$

By Art. 99, $$(a^{mc+nb})^{\frac{1}{nc}} = a^{\frac{m}{n}+\frac{b}{c}}.$$

Therefore, $$a^{\frac{m}{n}} \times a^{\frac{b}{c}} = a^{\frac{m}{n}+\frac{b}{c}}.$$

Similarly, $a^{\frac{m}{n}} \times a^{\frac{b}{c}} \times a^{\frac{r}{s}} \times \cdots\cdot a^{\frac{t}{u}} = a^{\frac{m}{n}+\frac{b}{c}+\frac{r}{s}+\cdots\cdot\frac{t}{u}}$. Hence,

I. *The product of several expressions consisting of the same factor, affected with any exponent, is the factor with an exponent equal to the sum of the exponents of the factors.*

By Arts. 101, 21, $a^{\frac{m}{n}} \div a^{\frac{b}{c}} = a^{\frac{m}{n}} \times a^{-\frac{b}{c}} = a^{\frac{m}{n}-\frac{b}{c}}$. Hence,

II. *The quotient of two expressions consisting of the same factor, affected with any exponent, is the factor with an exponent equal to that of the dividend minus that of the divisor.*

Illustrations: I. $5^{\frac{1}{2}} \times 5^{-\frac{3}{4}} \times 5 = 5^{\frac{1}{2}-\frac{3}{4}+1} = 5^{\frac{3}{4}} = \sqrt[4]{125}$; $x^{\frac{1}{2}} \times x^0 \times x^n = x^{n+\frac{1}{2}}$.

II. $2^{\frac{3}{4}} \div 2^{\frac{1}{2}} = 2^{\frac{3}{4}-\frac{1}{2}} = 2^{\frac{1}{4}} = \sqrt[4]{2}$; $(a+b)^{\frac{2}{3}} \div (a+b)^{\frac{3}{4}} = (a+b)^{\frac{2}{3}-\frac{3}{4}} = (a+b)^{-\frac{1}{12}}$.

Exercise 91.

Simplify:

1. $16^{-\frac{3}{4}} \times 16^{-\frac{1}{2}}$; $25^{-\frac{1}{2}} \times 25^{\frac{1}{2}}$; $x^{\frac{1}{2a}} \times x^{\frac{2}{a}}$; $a^{-\frac{1}{4}} \times \sqrt[4]{a}$.

2. $a^{\frac{n}{4}} \times a^{\frac{n}{3}} \times a^{\frac{5n}{12}}$; $n^{-2x} \times n^{-\frac{x}{y}}$; $m^{-\frac{2}{n}} \times m^{-\frac{m}{n}}$; $2^{-\frac{1}{2}}\sqrt{2}$.

3. $y^{-\frac{a}{c}} \times y^{-\frac{m}{n}}$; $a^{\frac{1}{x}} \div a^{-\frac{2}{x}}$; $a^{\frac{n}{4}} \times a^{\frac{n}{3}}$; $(a^6)^{\frac{1}{2}} \div (a^6)^{\frac{2}{3}}$.

4. $(-2)^{-\frac{1}{4}} \div (-32)^{-\frac{1}{4}}$; $a^{\frac{5}{6}} \div a^{\frac{2}{3}}$; $(x^{-2})^{-\frac{1}{3}} \div (xy^3)^{-\frac{1}{3}}$.

5. $x^{\frac{1}{a}} \div x^{\frac{3}{2a}}$; $a^{-\frac{x}{5}} \div a^{-\frac{2x}{5}}$; $(a-b)^{\frac{1}{2}} \div (a^{\frac{1}{2}} + b^{\frac{1}{2}})^{\frac{1}{2}}$.

6. $3^{2n} \div 3^n$; $(a-b)^n \div (a-b)$; $(x^{-1} y^{\frac{3}{2}})^{\frac{1}{3}} \div (y x^{\frac{1}{3}})^{\frac{1}{3}}$.

7. $\left(\frac{a^{-2} b}{a^3 b^{-4}}\right)^{\frac{3}{5}} \times \left(\frac{a b^{-1}}{a^{-3} b^2}\right)^{\frac{3}{5}}$; $\left(\frac{a^{-\frac{2}{3}} x^{\frac{1}{2}}}{a x^{-1}}\right)^2 \times \left(\frac{a^{-1}}{x^{-3}}\right)^{-\frac{1}{3}}$.

8. $(a-2x)^{\frac{1}{2}} \times (a-2x)^3 \times (2x-a)^5 \times (a-2x)^{\frac{17n}{2}}$.

9. $x^{m+n} \div x^{m-n}$; $(\sqrt{a} - \sqrt{b})^{\frac{m}{n}} (\sqrt{a} - \sqrt{b})^{-\frac{m}{n}}$.

10. $(x+y)^{m-n} \div (x+y)^{-n}$; $a^{3x+2y} \div a^{2x-3y}$; $n^{\frac{7}{12}} \div n^{\frac{3}{4}}$.

11. $\left(\frac{x^m}{a^p y^n}\right)^{\frac{1}{mnp}} \div \left(\frac{x^{m-1}}{a^{p+1} y^{n+1}}\right)^{\frac{1}{mnp}}$; $\left(\frac{a}{b}\right)^{\frac{m+2n}{n}} \times \left(\frac{a}{b}\right)^{\frac{n}{m}}$.

12. $(a+b)^{\frac{m}{n}} \times (a+b)^{\frac{n-1}{m-1}}$; $\left(\frac{5m^3}{3n}\right)^x \div \left(\frac{15m^2}{6n^3}\right)^x$.

13. $a^{\frac{5}{8}} \div a^{\frac{1}{2}}$; $2^n \times (2^n)^{n-1} \times 2^{n+1} \times 2^{n-1} \times 4^{-n}$.

14. $\frac{\sqrt[3]{a^2}}{\sqrt[3]{a^{-1}}}$; $\frac{(x^a)^3}{x^{b+c}} \times \frac{(x^b)^3}{x^{a+c}} \times \frac{(x^c)^3}{x^{a+b}}$; $\left(x^{\frac{1}{b}} y^{\frac{1}{a}}\right)^{ab} \div \left(\frac{x^{\frac{1}{a-b}}}{y^{\frac{1}{a+b}}}\right)^{a^2-b^2}$.

104. $$\left[\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}}\right]^{nq} = \left(a^{\frac{m}{n}}\right)^{pn} = a^{mp}.$$

Take the nqth root of the first and last members,

$$\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}} = a^{\frac{mp}{nq}}.$$

Similarly, $$\left(a^{\frac{m}{n}} b^{\frac{r}{s}} c^{\frac{t}{u}} \ldots\right)^{\frac{p}{q}} = a^{\frac{mp}{nq}} b^{\frac{rp}{sq}} c^{\frac{tp}{uq}} \ldots$$

The principles of this chapter are true, whatever the values of $a, b, c, \ldots m, n, p$, and q; that is, $a, b, c, \ldots m, n, p$, and q can be *positive* or *negative*, *integral* or *fractional*.

Illustrations. $(2^{\frac{3}{2}} \times 3^{\frac{1}{3}} \times 4^{-\frac{2}{3}})^{\frac{3}{2}} = 2^{\frac{3}{2} \times \frac{3}{2}} \times 3^{\frac{1}{3} \times \frac{3}{2}} \times 4^{-\frac{2}{3} \times \frac{3}{2}} = 2^2 \times 2^{\frac{1}{4}} \times 3^{\frac{1}{2}} \times 4^{-1} = \sqrt[4]{2} \times \sqrt{3}$; $\left\{\left[\left(a^{-\frac{2m}{n}}\right)^{-\frac{n}{m}}\right]^{\frac{p}{q}}\right\}^{\frac{q}{p}} \div \left\{\left[\left(a^{\frac{n}{m}}\right)^{\frac{m}{n}}\right]^{\frac{q}{p}}\right\}^{\frac{p}{q}}$ $= \left[\left(a^{-\frac{2m}{n}}\right)^{-\frac{n}{m}}\right]^{\frac{p}{q} \times \frac{q}{p}} \div \left[\left(a^{\frac{n}{m}}\right)^{\frac{m}{n}}\right]^{\frac{q}{p} \times \frac{p}{q}} = a^{-\frac{2m}{n} \times -\frac{n}{m}} \div a^{\frac{n}{m} \times \frac{m}{n}} = a^2 \div a = a.$

105. Negative and Fractional Root Indices.

$$\sqrt[-n]{a^m} = a^{\frac{m}{-n}} = a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} = \frac{1}{\sqrt[n]{a^m}}.$$

Similarly, $\sqrt[-\frac{c}{n}]{a^m} = a^{\frac{m}{-\frac{c}{n}}} = a^{-\frac{m}{\frac{c}{n}}} = \frac{1}{a^{\frac{m}{\frac{c}{n}}}} = \frac{1}{\sqrt[\frac{c}{n}]{a^m}}$. Hence,

A negative root index, either integral or fractional, indicates the reciprocal of the expression with a corresponding positive index.

Note. Since it is impossible to extract a fractional or negative root, or raise an expression to a fractional or negative power, in order to perform the operation indicated by such indices some preliminary transformations must be made.

Illustrations. $\sqrt[-\frac{3}{2}]{a^2} = \frac{1}{\sqrt[\frac{3}{2}]{a^2}} = \frac{1}{a^{\frac{2}{\frac{3}{2}}}} = \frac{1}{a^{\frac{4}{3}}} = \frac{1}{a\sqrt[3]{a}}$; $\sqrt[\frac{2}{3}]{4a^2} = (4a^2)^{\frac{1}{\frac{2}{3}}}$ $= (2^2 a^2)^{\frac{3}{2}} = 8a^3$; $\sqrt[-2]{4^{-2}} = 4^{\frac{-2}{-2}} = 4$; $\sqrt[-\frac{3}{5}]{27} = \frac{1}{\sqrt[\frac{3}{5}]{27}} = \frac{1}{(3^3)^{\frac{1}{\frac{3}{5}}}} = \frac{1}{(3^3)^{\frac{5}{3}}}$ $= \frac{1}{3^5} = \frac{1}{243}$; $\frac{1}{\sqrt[\frac{c}{n}]{a^m}} = \frac{1}{a^{\frac{m}{\frac{c}{n}}}} = \frac{1}{a^{\frac{mn}{c}}} = \frac{1}{\sqrt[c]{a^{mn}}}.$

Exercise 92.

Simplify:

1. $\sqrt[\frac{3}{2}]{27}$; $\sqrt[-4]{16}$; $\sqrt[\frac{5}{2}]{32\,m^{-10}}$; $\sqrt[-\frac{3}{2}]{3^{\frac{3}{8}}\,a^{-3}}$; $\sqrt[\frac{1}{4}]{x^{-\frac{3}{4}}}$.

2. $\sqrt[\frac{3}{4}]{8}$; $[(b^3)^2(a^4)^3(b^{-3})(a^{-5}b^{-1})^2]^5$; $\sqrt[\frac{1}{3}]{8\,a^n b^{n-1} c^{n-2}}$.

3. $(7\,a^2\,x^{2-2}\,y^{m-1})^3$; $\sqrt[\frac{1}{3m}]{3\,a^{\frac{3}{2}}\,b^{\frac{n-2}{m}}}$; $(x^{18n} \times x^{-12})^{\frac{1}{3n-2}}$.

4. $\left(\dfrac{16\,m^{-4}}{81\,n^3}\right)^{-\frac{3}{4}}$; $\left(\dfrac{9\,m^4}{16\,n^{-3}}\right)^{-\frac{3}{2}}$; $\left(\dfrac{256}{625}\right)^{-\frac{3}{4}}$; $(64\,x^{18})^{-\frac{5}{6}}$.

5. $\dfrac{(a^{-2}\,b)^3}{a^3\,b^{-4}} \div \left(\dfrac{a\,b^4}{a^3\,b^2}\right)^5$; $(x^{-a}\,y^{-b})^{-3}\,(x^3\,y^2)^{-a}$; $\sqrt[-\frac{3}{2}]{\dfrac{1}{125}}$.

6. $\sqrt[-\frac{3}{2}]{27\,n^{-3}}$; $(4\,x^{-\frac{2}{3}})^{-\frac{3}{2}}$; $(a\,b^{-1}\,c^{-2})^{\frac{1}{3}}\,(a^{-1}\,b^{-2}\,c^{-4})^{-\frac{1}{6}}$.

7. $\left(\dfrac{m\,y}{x}\right)^{\frac{1}{2}}\left(\dfrac{n\,x}{y^2}\right)^{\frac{1}{3}}\left(\dfrac{y^2}{m^2\,n^2}\right)^{\frac{1}{4}}$; $\sqrt[\frac{1}{2mn}]{5\,a^{\frac{n-1}{2m}}\,b^{\frac{2}{n}}\,c^{\frac{m^{\frac{1}{2}}}{n^{\frac{1}{2}}}}}$; $\sqrt[-\frac{2}{3}]{\dfrac{16}{25}}$.

8. $\sqrt[\frac{1}{m}]{\dfrac{x^{m+n}}{x^n}} \div \sqrt[\frac{1}{m-n}]{\dfrac{x^n}{x^{n-m}}}$; $\left(x^{-\frac{a}{b}}\,y^{-1}\right)^b \div \dfrac{\left(x^{\frac{1}{a+b}}\right)^{a^2-b^2}}{\left(y^{\frac{1}{a+b}}\right)^{ab+b^2}}$.

Queries. What does a negative exponent indicate? A fractional exponent? A negative fractional root index? Any expression with 0 for its exponent = ? Why? What is the product of $a^{\frac{3}{5}}$ and $a^{\frac{2}{3}}$? Prove it.

Miscellaneous Exercise 93.

Express with fractional exponents and negative power indices:

1. $\sqrt[-\frac{2}{3}]{a^{-2}}$; $\sqrt[-2]{a^m}$; $\sqrt[\frac{3}{2}]{a^n}$; $\sqrt[-7]{a^6}$; $(\sqrt[3]{a})^7$; $\sqrt[3]{a^5\,b^{-2}}$.

2. $\sqrt[-3]{ab^2c^4}$; $\sqrt[5]{a^3b^2c^4}$; $\sqrt[-\frac{m}{n}]{a^xb^c}$; $\sqrt[20]{x^{-a}y^{bc}}$; $5\sqrt[6]{a^2b\,c^{-3}x^{-5}y^4}$.

Express with radical signs and negative integral root indices:

3. $a^{-\frac{2}{3}}$; $a^{\frac{1}{3}}\,b^{\frac{1}{4}}\,c^{-\frac{1}{5}}$; $4\,ab^{-\frac{2}{3}}$; $7\,a^{-\frac{1}{2}}\,x^{-\frac{m}{n}}$; $\dfrac{3}{x^{-\frac{3}{4}}}$.

Express with radical signs and fractional root indices:

4. $a^{\frac{2}{\frac{2}{3}}}$; $(4\,a^2)^{\frac{1}{\frac{3}{2}}}$; $a^{\frac{m}{\frac{1}{2}}}\,b^{\frac{n}{\frac{1}{3}}}\,c^{\frac{x}{\frac{1}{4}}}$; $a^{\frac{\frac{1}{n}}{\frac{1}{m}}}\,b^{\frac{\frac{m}{n}}{\frac{1}{n}}}$; $a^{\frac{m}{n}}\,b^{m}$; $x\,y$.

Express with fractional exponents and fractional power indices:

5. $\sqrt[\frac{1}{3}]{a^{\frac{5}{3}}\,b^{\frac{3}{4}}}$; $\sqrt[\frac{m}{n}]{2^{\frac{x}{a}}\,b^{\frac{x}{a}}}$; $3\sqrt[\frac{1}{5}]{(8a^{-3})^{-\frac{4}{3}}}$; $\sqrt[\frac{1}{n}]{a^{-\frac{1}{n}}}$; $\sqrt[\frac{1}{m}]{a}$; $\sqrt[-a]{5}$.

Express in the form of integral expressions:

6. $\dfrac{3\,a^2 b}{c^{-2}}$; $\dfrac{5}{a\,b\,c}$; $\dfrac{m^{\frac{2}{3}}}{n^{-\frac{3}{2}}}$; $\dfrac{a}{\sqrt[\frac{3}{2}]{b^{-1}}}$; $\dfrac{x^{-1}}{\sqrt[3]{y^{\frac{1}{3}}}}$; $\dfrac{x^{-\frac{1}{3}}}{\sqrt[-\frac{2}{3}]{y^{-\frac{3}{2}}}}$; $\dfrac{a^2}{a^{\frac{1}{3}}\,b^{-\frac{2}{3}}}$.

Express with literal factors transposed from the numerators to the denominators:

7. $\dfrac{2\,a^2 c^{-1}}{5\,x}$; $a^{\frac{3}{2}}\,b^{-\frac{2}{3}}$; $\sqrt[n]{a\,b^m}$; $\dfrac{x^{-1}y^{\frac{2}{5}}}{z^2}$; $\dfrac{5\,a^{-2}\,b^{-\frac{3n}{m}}}{3\,c^m}$; $\sqrt[\frac{1}{n}]{x^{-2m}\,y^{-1}}$.

Simplify and express with positive exponents:

8. $\sqrt[\frac{2}{3}]{a^2}$; $\sqrt[-\frac{3}{4}]{8\,a^{-3}}$; $\sqrt[\frac{1}{n}]{a^{\frac{1}{n}}\,b^{\frac{m}{n}}}$; $\sqrt[-\frac{2}{3}]{a^{-2}}$; $a^{\frac{2}{-3}}$; $[\sqrt{a^3} \div \sqrt{a^5}]^{\frac{1}{-3}}$.

9. $\dfrac{2\,a^{\frac{1}{2}} \times 3\,a^{-1}}{\sqrt{a^3}}$; $\dfrac{a\,b^2 \times a^{-1} \times \sqrt[4]{a^3}}{\sqrt{b^{-3}}}$; $\sqrt[5]{m^{-3}} \div \sqrt[5]{m^7}$.

10. $x^{-\frac{1}{3}} \times 2\,x^{-\frac{1}{2}}$; $\left(\dfrac{16\,x^2}{y^{-2}}\right)^{-\frac{1}{4}}$; $a^0 \times a^{\frac{1}{2}} \times a^{-\frac{1}{2}}$.

11. $\sqrt[-2]{4\,a^{-4}}$; $\left(\dfrac{27\,a^3}{8\,n^{-3}}\right)^{-\frac{2}{3}}$; $\sqrt{a^{-2}\,b} \times \sqrt[3]{a\,b^{-3}}$.

12. $a^{\frac{5}{8}}\,b^{\frac{1}{4}}\,c^{\frac{1}{2}} \times a^{-\frac{1}{8}}\,b^{-\frac{1}{2}}\,c^{-\frac{1}{2}}$; $\left(\dfrac{m^{-\frac{1}{2}}}{4\,a^2}\right)^{-2}$; $\sqrt[\frac{1}{2}]{a^{-\frac{2}{a}}}$.

13. $a^{\frac{1}{3}}b^{\frac{2}{3}}c^{\frac{1}{6}} \times a^{-\frac{2}{3}}b^{-\frac{1}{2}}c^{-\frac{1}{2}}$; $\sqrt[-3]{\dfrac{a^{-9}}{27}}$; $\sqrt[-6]{(x^{-\frac{2}{3}}y^{\frac{1}{2}})^3}$.

14. $\sqrt[\frac{1-n}{n}]{x\sqrt[n]{x^{-\frac{1}{n}}}}$; $(x+y)^0 \times n^{\frac{3}{4}} \times n^{-\frac{1}{12}} \times \sqrt{n}$.

15. $\sqrt[-\frac{4}{x}]{(m\sqrt[3]{m^{-1}})^{-\frac{1}{x}}}$; $\sqrt[-\frac{m}{n}]{a^{\frac{1}{n}}b^{\frac{1}{n}}}$; $\dfrac{1}{m}\left(\dfrac{a}{m}\right)^{-1}$; $\sqrt[-\frac{2}{3}]{a^{-n}}$.

16. $(m^{-\frac{1}{2}}\sqrt[3]{a})^{-3} \times \sqrt[\frac{1}{n}]{(a^{-2}\sqrt{m^{-6}})^n}$; $\sqrt[\frac{1}{n^2-1}]{x^{\frac{1}{n+1}}} + \dfrac{\sqrt[2n]{x^{2n^2}}}{x}$.

17. $\sqrt[n]{a^{n+k}b^{2n-k}} \div \left(a^{\frac{1}{n}}b^{-\frac{1}{n}}\right)^k$; $\dfrac{3x^0m^{-2}n^{-4}}{5x^{-1}m^{-3}n^{-5}}$.

18. $\sqrt[3]{(x+y)^5} \times (x+y)^{-\frac{2}{3}}$; $\dfrac{x^{-n}}{y^0}$; $\sqrt[\frac{1}{n}]{(x-y)^{-3}} \div \sqrt[\frac{1}{3}]{(x+y)^n}$.

19. $\dfrac{4(x^0+y^0)^{-2} \times m^{-5}}{5m^{-3}n^{-4}}$; $\sqrt[\frac{1}{6}]{\dfrac{\sqrt[3]{a}}{\sqrt[4]{b^{-1}}} \times \left(\dfrac{b^{\frac{1}{4}}}{a^{\frac{1}{3}}}\right)^2 \div \dfrac{a^{-\frac{1}{3}}}{b^{-\frac{1}{2}}}}$.

20. $\dfrac{8x^{-5}}{m^2} \times \dfrac{(m^0+n^0)^{-3}}{m^{-10}}$; $\left(\dfrac{4^{-\frac{3}{4}} - \dfrac{1}{2^{-3}}}{\dfrac{1}{3^{-2}} + 4^{-3}}\right)^0$; $\dfrac{(m+n)^2(m-n)^2}{(m^2-n^2)^2}$.

21. $\dfrac{(m^{\frac{1}{2}}+n^{\frac{1}{2}})^{\frac{1}{2}}(m^{\frac{1}{2}}-n^{\frac{1}{2}})^{\frac{1}{2}}}{(m-n)^{\frac{3}{2}}}$; $\left(\dfrac{a^{-3m}+b^{-2n}}{a^{-6m}-b^{-4n}}\right)^{-3}$.

22. $\dfrac{2^n(2^n-1)^n}{2^{n+1} \times 2^{n-1}} \times \dfrac{1}{4^{-n}}$; $\dfrac{2^{n+1}}{(2^{2n})^{n-1}} \div \dfrac{4^{n+1}}{(2^n-1)^{n+1}}$.

23. $\dfrac{\left(9^n \times 3^2 \times \dfrac{1}{3^{-n}}\right) - 27^n}{3^{3n} \times 9}$; $\left(2^{\frac{n}{m}} \times 3^{\frac{1}{mn}}\right)^{mn}\left(5^{\frac{n}{m}} \times 6^{\frac{1}{mn}}\right)^{mn}$.

Multiply:

24. $a^{\frac{m}{n}}b^{-\frac{t}{s}} - a^{\frac{m}{2n}}b^{-\frac{t}{2s}} + 1$ by $a^{\frac{m}{2n}}b^{-\frac{t}{2s}} + 1$.

25. $a^{\frac{m}{n}} + a^{\frac{m}{2n}}b^{\frac{t}{4s}} + b^{\frac{t}{2s}}$ by $a^{\frac{m}{n}} - a^{\frac{m}{2n}}b^{\frac{t}{4s}} + b^{\frac{t}{2s}}$.

26. $a^{\frac{3m}{n}}b^{-\frac{3t}{s}} - a^{\frac{m}{n}}b^{-\frac{t}{s}} + a^{-\frac{m}{n}}b^{\frac{t}{s}} - a^{-\frac{3m}{n}}b^{\frac{3t}{s}}$ by $a^{\frac{m}{n}}b^{-\frac{t}{s}} + a^{-\frac{m}{n}}b^{\frac{t}{s}}$.

Divide:

27. $a^{\frac{2m}{n}} + a^{\frac{m}{n}}b^{\frac{t}{2s}} + b^{\frac{t}{s}}$ by $a^{\frac{m}{n}} + a^{\frac{m}{2n}}b^{\frac{t}{4s}} + b^{\frac{t}{2s}}$.

28. $x^{2n(n-1)} - y^{2m(m-1)}$ by $x^{n(n-1)} \pm y^{m(m-1)}$.

29. $x^{4n-4} - y^{4m-4}$ by $x^{2n-2} + y^{2m-2}$.

30. $x^{4n^2-4n} - y^{2m^2-2m}$ by $x^{2n^2-2n} \pm y^{m^2-m}$.

31. $a^{\frac{m}{n}} - x^{\frac{t}{s}} + 4a^{\frac{m}{4n}}x^{\frac{3t}{4s}} - 4a^{\frac{m}{2n}}x^{\frac{t}{2s}}$ by $a^{\frac{m}{2n}} + 2a^{\frac{m}{4n}}x^{\frac{t}{4s}} - x^{\frac{t}{2s}}$.

32. $a^{\frac{3n}{2}} + a^{-\frac{3}{2}n}$ by $a^{\frac{n}{2}} + a^{-\frac{n}{2}}$; $ny^{\frac{7}{3}} + mx^{\frac{7}{5}}$ by $n^{\frac{1}{7}}y^{\frac{1}{3}} + m^{\frac{1}{7}}x^{\frac{1}{5}}$.

Separate into two factors:

33. $a^{-1} - b$; $a^{-\frac{3}{7}} - b^{-\frac{3}{7}}$; $a^{\frac{2n}{5}} - b^{-2n}$.

Expand:

34. $(a^{-1}b - b^{-1}x)^4$; $(x - 2x^{-1})^3$; $\left[\left(a^{-\frac{n}{2}} - a^{\frac{n}{2}}\right)^{\frac{5}{n}}\right]^n$.

Resolve into prime factors, and find the products of:

35. $\sqrt[3]{12}$, $\sqrt[4]{72}$, $\sqrt[6]{96}$, $\sqrt[8]{64}$.

36. $\sqrt[3]{12}$, $\sqrt[4]{72}$, $\sqrt[6]{96}$, $\sqrt[8]{64}$, $\sqrt[4]{576}$, $\sqrt{24}$.

Find the cube roots of:

37. $a^{\frac{3}{2}}+\frac{3}{2}a+\frac{3}{4}a^{\frac{1}{2}}+\frac{1}{8}$; $8b^2-12a^{-1}b^{\frac{11}{6}}+6a^{-2}b^{\frac{5}{3}}-a^{-3}b^{\frac{3}{2}}$.

38. $8a^{-2}-12a^{-\frac{11}{6}}+6a^{-\frac{5}{3}}-a^{-\frac{3}{2}}$.

39. $x^3-9x+27x^{-1}-27x^{-3}$.

Find the 6th roots of:

40. $x^6+\dfrac{1}{x^6}-6\left(x^4+\dfrac{1}{x^4}\right)+15\left(x^2+\dfrac{1}{x^2}\right)-20$.

41. $729-2916a^{2n}+4860a^{4n}-4320a^{6n}+2160a^{8n}$
$-576a^{10n}+64a^{12n}$.

42. $x^{-12}-6x^{-10}+15x^{-8}-20x^{-6}+15x^{-4}-6x^{-2}+1$.

Simplify and express with positive exponent:

43. $\sqrt[\frac{n+1}{n}]{a^{n^2-1}}+\sqrt[\frac{1}{n}]{\dfrac{a^{\frac{2}{n}}}{a^{\frac{1}{n}}}}$; $\dfrac{\sqrt[\frac{1}{n}]{4\times 4^{n-1}}}{\sqrt[\frac{n+1}{n-1}]{4^{\frac{(n+1)^2}{n-1}}\times 4^{n+1}}}\div\sqrt[-\frac{1}{n}]{16}$.

44. $\sqrt[-\frac{5}{n}]{\left(\dfrac{x^{-2}y^3}{x^3y^{-2}}\right)^{\frac{1}{n}}}\times\sqrt[\frac{1}{n}]{\left(\dfrac{x^{-3}y^3}{x^3y^{-3}}\right)^{-\frac{1}{n}}}$; $[(-x^{-n})^{-2m}]^{-a}$.

45. $\left(\dfrac{m^{-3}}{n^{-\frac{2}{3}}x}\right)^{-\frac{3}{2}}\div\left(\dfrac{\sqrt{m^{-\frac{1}{2}}}\times\sqrt[6]{n^3}}{m^2x^{-1}}\right)^{-2}$; $x^{-0}+y^{-0}$.

46. $\dfrac{a^0b^{-2}}{9(x^0+y^0+z^0)^{-2}m^3}$; $\dfrac{[(8a-6b)^{2n}]^{5n}}{[(4a-3b)^{5n}]^{2n}}$; x^0+y^0.

47. $\dfrac{(20x+8x^2y^2-12y^4)^n}{[4(x^2+y^2)]^n}$; $\dfrac{(m^3+n^3)^2(m^3-n^3)^2}{m^6-n^6}$.

CHAPTER XIX.

RADICAL EXPRESSIONS.

106. A **Surd** is an indicated root that cannot be exactly obtained; as, $\sqrt{5}$; $\sqrt[3]{\frac{2}{3}}$; $\sqrt[4]{a^3}$.

The **Order** of a surd is indicated by the root index.

Surds are said to be of the *second*, *third*, *fourth*, etc., or nth order, according as the *second*, *third*, *fourth*, etc., or nth roots are required. Thus, $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[4]{5}$, etc., $\sqrt[n]{x}$, are *quadratic*, *cubic*, *biquadratic*, etc.

Surds are of the *same order* when they have the same root index; as, $\sqrt[4]{5}$, $\sqrt[4]{a^2}$, and $\sqrt[4]{b^n}$.

A surd is in its *simplest form* when the expression under the radical sign is *integral*, and in the *lowest degree possible;* as, $\sqrt[3]{32\,a^4} = \sqrt[3]{2^3\,a^3 \times 4\,a} = 2\,a\sqrt[3]{4\,a}$.

Similar or **Like Surds** are those which, when reduced to their simplest forms, have the *same surd factor;* as, $3\sqrt{3}$ and $\sqrt{3}$; $2\,a\sqrt[5]{b}$ and $c\sqrt[5]{b}$. Otherwise the surds are *dissimilar.*

Notes: 1. When a surd is expressed by means of the radical sign, it is called a **Radical Expression.**

2. An **Irrational Expression** is one which involves a surd; as, $\sqrt{3}$; $a + b\sqrt[n]{c^m}$.

3. An indicated root may have the *form* of a surd, without really being a surd. Thus, $\sqrt{4}$ and $\sqrt[3]{a^9}$ have the *form* of surds.

4. *Rational factors* or *expressions* are those which are not surds; as, 2; $a^2x - b^2y$.

5. Since $a^{\frac{m}{n}} = a^{\frac{mp}{np}}$, surds of the form $\sqrt[n]{a^m}$ and $\sqrt[np]{a^{mp}}$ are equivalent surds of different orders.

6. A **Mixed Surd** is the product of a rational factor and a surd factor; as, $a\sqrt{b}$; $3\sqrt{5}$.

7. An **Entire Surd** is one in which there is no rational factor outside of the radical sign; as, $\sqrt{2}$; $\sqrt[3]{a^4}$; $\sqrt[n]{x}$.

8. A *binomial surd* has two terms, and involves one or two surds; as, $a+b\sqrt[n]{x}$; $a\sqrt[n]{x}-b\sqrt[m]{y}$. A *compound surd* or *polynomial* has two or more terms, and involves one or more surds; as, $\sqrt[2]{2}+3\sqrt[3]{4}-5\sqrt[4]{3}$; $a+b-c+2\sqrt{a}$.

9. Quadratic surds are of most frequent occurrence.

107. The methods for operating with surds follow from an application of the principles of Chapter XVIII. Thus,

$$\tfrac{2}{3}=\sqrt{\tfrac{4}{9}}.\quad 2a^2b^3=\sqrt[3]{(2a^2b^3)^3}=\sqrt[3]{8a^6b^9}.\quad \text{In general,}$$
$$a=a^{\frac{1}{1}}=a^{\frac{n}{n}}=\sqrt[n]{a^n}.\quad \text{Hence,}$$

I. **To Reduce a Rational Factor to the Form of a Surd of any Order.** Raise it to the power indicated by the root index, and place it under the radical sign.

$$2\sqrt{3}=\sqrt{2^2\times 3}=\sqrt{12}.\quad \tfrac{2}{3}\sqrt[3]{9}=\sqrt[3]{(\tfrac{2}{3})^3\times 9}=\sqrt[3]{\tfrac{8}{3}}.\quad \text{In general,}$$
$$a\sqrt[n]{x}=a^{\frac{n}{n}}x^{\frac{1}{n}}=(a^n x)^{\frac{1}{n}}=\sqrt[n]{a^n x}.\quad \text{Hence,}$$

II. **To Change a Mixed Surd to the Form of an Entire Surd.** Reduce the rational factor to the form of the surd, multiply by the surd factor, and place the product under the radical sign.

$$\sqrt{72}=\sqrt{6^2\times 2}=6\sqrt{2}.\quad \sqrt[3]{1029\,a^4}=(7^3a^3\times 3a)^{\frac{1}{3}}=7a\sqrt[3]{3a}.$$
$$2\sqrt[3]{\tfrac{1}{2}}=2\sqrt[3]{\frac{1\times 4}{2\times 4}}=\frac{2\sqrt[3]{4}}{2}=\sqrt[3]{4}.\quad \frac{a}{2}\sqrt[4]{\frac{1}{2a^3}}=\frac{a}{2}\sqrt[4]{\frac{1\times 2^3a}{2a^3\times 2^3a}}$$
$$=\frac{a}{2}\sqrt[4]{\frac{8a}{2^4a^4}}=\tfrac{1}{4}\sqrt[4]{8a}.\quad \text{In general,}$$
$$\sqrt[n]{x^n y}=xy^{\frac{1}{n}}=x\sqrt[n]{y}.\quad \sqrt[n]{\frac{y}{x^m}}=\sqrt[n]{\frac{y\times x^{n-m}}{x^m\times x^{n-m}}}=\sqrt[n]{\frac{yx^{n-m}}{x^n}}=\frac{1}{x}\sqrt[n]{yx^{n-m}}.$$

Hence,

III. **To Reduce a Surd to its Simplest Form.** If the surd is integral, remove from under the radical sign all factors of which the indicated root can be exactly obtained.

If the surd is fractional, multiply its numerator and denominator by such expression that the indicated root of the denominator can be exactly obtained.

$$\sqrt{\tfrac{1}{2}} \times \sqrt{\tfrac{3}{4}} \times \sqrt{8} = \sqrt{\tfrac{1}{2} \times \tfrac{3}{4} \times 8} = \sqrt{3}.$$

$$\sqrt[3]{2^{\frac{3}{2}} a} \times \sqrt[3]{a^2} = \sqrt[3]{2^{\frac{3}{2}} a \times a^2} = a\sqrt{2}. \quad \text{In general,}$$

$$\sqrt[n]{a} \times \sqrt[n]{b} \times \sqrt[n]{c} \times \ldots . \sqrt[n]{p} = a^{\frac{1}{n}} \times b^{\frac{1}{n}} \times c^{\frac{1}{n}} \times \ldots . p^{\frac{1}{n}} = (abc \ldots . p)^{\frac{1}{n}}$$
$$= \sqrt[n]{abc \ldots . p}. \quad \text{Hence,}$$

IV. **To Find the Product of Two or More Surds of the Same Order.** Take the product of the expressions under the radical signs and retain the root index.

$$\sqrt{28} \div \sqrt{7} = \sqrt{\tfrac{28}{7}} = 2. \quad \sqrt[3]{7} \div \sqrt[3]{56} = \sqrt[3]{\tfrac{7}{56}} = \sqrt[3]{\tfrac{1}{8}} = \tfrac{1}{2}.$$

In general,

$$\sqrt[n]{x^m} \div \sqrt[n]{y^a} = \left(\frac{x^m}{y^a}\right)^{\frac{1}{n}} = \sqrt[n]{\frac{x^m}{y^a}}. \quad \text{Hence,}$$

V. **To Find the Quotient of Two Surds of the Same Order.** Take the quotient of the expressions under the radical signs and retain the root index.

$$\sqrt{\sqrt[3]{64}} = \sqrt[6]{64} = 2. \quad \sqrt[2n]{\sqrt{256^n}} = \sqrt[4n]{(2^8)^n} = 2^2 = 4. \quad \text{In general,}$$

$$\sqrt[m]{\sqrt[n]{a}} = \left(a^{\frac{1}{n}}\right)^{\frac{1}{m}} = a^{\frac{1}{mn}} = \sqrt[mn]{a}. \quad \text{Hence,}$$

VI. **To Find the mth Root of the nth Root of an Expression.** Take the mnth root of the expression.

Note. It is sometimes easier to perform operations with surds if the arithmetical numbers contained in the surds be *expressed in their prime factors,* and *fractional exponents* be used instead of radical signs.

Exercise 94.

Express in the form of surds of the 3d and nth orders, respectively:

1. $\frac{1}{2}$; $\frac{3}{2}$; 2^2; 4^n; $2\,a^n$; $3\,a\,b\,c$; $3\,x$; a^3; x^x; $x^n\,y^n$.

Express as entire surds:

2. $\frac{1}{2}\sqrt{2}$; $\frac{1}{3}\sqrt[3]{3}$; $5\sqrt{32}$; $\frac{2}{7}\sqrt{\frac{91}{8}}$; $16\sqrt{\frac{275}{288}}$; $a\,b\sqrt{b\,c}$.

3. $a\sqrt[3]{a^2\,b\,c^3}$; $3\,a^2\sqrt[4]{a\,b^3}$; $\frac{1}{2}\sqrt[4]{x^2}$; $2\,x\sqrt[5]{x\,y}$; $\frac{5}{2}\sqrt[3]{9\frac{3}{5}}$.

4. $\dfrac{3}{2\,x}\sqrt[3]{\dfrac{9}{4\,x^2}}$; $a^2\sqrt[3]{a\,b\,c}$; $3\,a\,b\,c\sqrt{a\,b\,c^{-1}}$; $x\sqrt[n]{\dfrac{a^2}{x^{n-2}}}$.

5. $5\,x\sqrt[n]{25\,x^{-1}}$; $(m-1)\sqrt{\dfrac{m+1}{m-1}}$; $\dfrac{m+n}{m-n}\sqrt{\dfrac{m-n}{m+n}}$.

6. $(a+b)\sqrt{\dfrac{a\,b}{a^2+2\,a\,b+b^2}}$; $(x+y)\sqrt{1-\dfrac{x-y}{x+y}}$.

7. $\dfrac{a\,x}{a-x}\sqrt{\dfrac{a^2-x^2}{a\,x}}$; $\dfrac{x}{y}\sqrt[m]{\dfrac{y^{m+1}}{x^{m-1}}}$; $\dfrac{y}{x^n}\sqrt{\dfrac{x^{2n+1}}{y^3}}$; $6\,a\sqrt[3]{\dfrac{n}{3\,a}}$.

Express in their simplest forms:

8. $\sqrt{288}$; $3\sqrt{150}$; $\sqrt[3]{-2187}$; $\frac{3}{5}\sqrt{90\frac{5}{8}}$; $2\sqrt[4]{80\,a^5}$.

9. $\sqrt{3\frac{1}{8}}$; $\sqrt[3]{1\frac{11}{16}}$; $\sqrt[3]{\frac{4}{25}}$; $\sqrt[3]{1029}$; $\sqrt{\frac{5}{9}}$; $\sqrt{\frac{5}{12}}$; $\sqrt[6]{\frac{1}{2}}$.

10. $\sqrt[5]{\dfrac{3}{4}}$; $\dfrac{12}{\sqrt{5}}$; $\sqrt[3]{-108\,a^4\,b^3}$; $\sqrt[5]{352\,a^{\frac{5}{6}}\,b^{10}}$; $\sqrt{\dfrac{3\,m^2\,n\,x}{4\,a\,b^3}}$.

11. $\sqrt[5]{a^{10}\,b^{-5}\,c^{5n+1}}$; $\dfrac{m^3\,n^2}{a^5}\sqrt{\dfrac{a^{10}}{m^5\,n^5}}$; $\dfrac{m^3\,n^2}{x^4}\sqrt{\dfrac{n^3\,x^9}{m}}$.

12. $\dfrac{1}{\sqrt[5]{586}}$; $\sqrt[3]{7290\, a^{3n}\, b^{6m+2}}$; $\sqrt[5]{\dfrac{2\, a\, x^2}{y}}$; $\sqrt[m]{x^{m+n}\, y^{2m}}$.

13. $\sqrt{(x+y)\,(x^2-y^2)}$; $\sqrt{a\, x^2 - 8\, a\, x + 16\, a}$.

14. $\dfrac{x}{x^2-y^2}\sqrt{\dfrac{x^2\, y - 2\, x\, y + x\, y^3}{y^2}}$; $\sqrt[3]{1715\, x^{9n-m}\, y^{6m}}$.

Simplify:

15. $\sqrt{12} \times \sqrt{18} \times \sqrt{24}$; $\sqrt{54} \div \sqrt{6}$; $\sqrt[3]{\sqrt{64}}$.

16. $\sqrt[3]{16} \times \sqrt[3]{-54} \times \sqrt[3]{128}$; $[\sqrt[3]{128\, a} \div \sqrt[3]{6\, a^2}] \div \sqrt[3]{9\, a^2}$.

17. $\sqrt[n]{\sqrt{\dfrac{9^n}{4^n}\, a^{2n}}} \times \sqrt{50\, a^3\, b^5} \div \sqrt{32\, a\, b^3}$.

18. $\sqrt[10]{2^5\, a^3\, m^5} \times \sqrt[10]{10^5\, a^2\, m^2\, x^5} \times \sqrt[10]{5^5\, a^5\, m^3\, x^5}$.

19. $(\sqrt[6]{5^3\, a^6\, b^9} \div \sqrt[6]{25\, a^4\, b^6}) \times \sqrt[6]{125\, a^3\, b} \times \sqrt[6]{5^2\, a\, b^2}$.

20. $(\sqrt{b^5\, c^6} \div \sqrt{b^3\, c^4}) \times \sqrt[3]{54\, a^4} \div \sqrt[3]{16\, a}$; $\sqrt[n]{a^{m+n}\, x}$.

21. $(\sqrt[3]{a\, b\, c^{-1}} \times \sqrt[3]{a^{-1}\, b^{-1}\, c}) \div (\sqrt[n]{x^{-10}\, y^{10}} \times \sqrt[n]{x^{10}\, y^{-10}})$.

22. $(\sqrt[3]{\tfrac{3}{4}} \div \sqrt[3]{\tfrac{8}{3}})\, \sqrt[6]{\sqrt{20736}}$; $\sqrt[2n]{4^n\, a^{-n}} \div \sqrt[2n]{a^n}$.

23. $\sqrt{\tfrac{2}{5}\, a^3} \times \sqrt{\tfrac{8}{3}\, a^{-2}} \times \sqrt{\tfrac{3}{2}\, a^{\frac{3}{2}}} \times \sqrt{2.5\, a^{-\frac{1}{2}}}$.

24. $\sqrt[m]{\sqrt[n]{\sqrt[p]{2^{2mn}\, a^{mnp}}}}$; $(16\, a^6\, b^2)^{\frac{1}{2}} \times (a^{\frac{1}{3}}\, b^{\frac{1}{2}})^2 \div \sqrt{2\, a^{\frac{2}{3}}\, b}$.

108. $\sqrt[3]{2} = \sqrt[3 \times \frac{9}{3}]{2^{1 \times \frac{9}{3}}} = \sqrt[9]{8}$. $3\sqrt[5]{2^2} = 3\sqrt[5 \times \frac{10}{5}]{2^{2 \times \frac{10}{5}}} = 3\sqrt[10]{16}$.

In general,

$$\sqrt[n]{a^p} = a^{\frac{p}{n}}\ (n > p) = a^{\frac{p \times m}{n \times m}} = \sqrt[nm]{a^{pm}}.\quad \text{Hence,}$$

I. To Reduce a Surd, in its Simplest Form, to an Equivalent Surd of a Different Order. Divide the required root index by the root index of the surd, and multiply the power and root index by the quotient.

$$\sqrt[3]{2} = \sqrt[3 \times \frac{18}{3}]{2^{1 \times \frac{18}{3}}} = \sqrt[3 \times 6]{2^{1 \times 6}} = \sqrt[18]{64},$$

$$\sqrt[9]{3^2} = \sqrt[9 \times \frac{18}{9}]{3^{2 \times \frac{18}{9}}} = \sqrt[18]{3^{2 \times 2}} = \sqrt[18]{81},$$

$$\sqrt[6]{5^{\frac{1}{3}}} = \sqrt[6 \times \frac{18}{6}]{5^{\frac{1}{3} \times \frac{18}{6}}} = \sqrt[18]{5^{\frac{1}{3} \times 3}} = \sqrt[18]{5},$$

The L.C.M. of the root indices (3, 9, 6) is 18.

In general,

$$\sqrt[n]{a^p} = a^{\frac{pm}{nm}} \; (n > p) = \sqrt[nm]{a^{pm}}.$$

$$\sqrt[m]{b^{p_1}} = b^{\frac{p_1 n}{mn}} \; (m > p_1) = \sqrt[nm]{b^{p_1 n}}.$$ Hence,

II. To Reduce Surds, in their Simplest Forms, to Equivalent Surds of the Same Lowest Order. Divide the L.C.M. of the indices by each index in succession. Multiply the power and root index of the first surd by the first quotient, of the second surd by the second quotient, and so on.

Exercise 95.

Express as surds of the 12th order:

1. $\sqrt[4]{2}$; $\sqrt[6]{3}$; $\frac{3}{2}\sqrt[3]{6}$; $3\sqrt{2}$; $\sqrt[6]{a^4}$; $\sqrt[8]{4}$; $\frac{1}{2}\sqrt[9]{8}$.

2. $\sqrt[4]{3^{\frac{1}{3}}}$; $\frac{4}{5}\sqrt[10]{32}$; $\sqrt[8]{a^{14}}$; $\sqrt[3]{a\,x^3} \times \sqrt[3]{a^{-1}x^{-1}}$.

Express as surds of the nth order, with positive exponents:

3. $\sqrt[3]{x^2}$; $\sqrt[3]{x^n}$; $a^{\frac{1}{2}}$; $\sqrt[6]{x^n\,y^{\frac{1}{n}}}$; $\frac{1}{a^{-1}}$; $\sqrt{a^{-\frac{1}{n}}}$; $\frac{\sqrt{a}}{x^{-n}}$.

Reduce the following to equivalent surds of the same lowest order:

4. $\sqrt{5},\ \sqrt[3]{11},\ \sqrt[6]{13};\ \ \sqrt{2},\ \sqrt[5]{5},\ \sqrt[4]{3};\ \ \sqrt[4]{8},\ \sqrt{3},\ \sqrt[8]{6}.$

5. $\sqrt[3]{2},\ \sqrt[9]{8},\ \sqrt[6]{4};\ \ \sqrt[6]{7},\ \sqrt[3]{5},\ \sqrt[4]{6};\ \ \sqrt{a},\ \sqrt[9]{a^5}.$

6. $\sqrt[5]{a^3},\ \sqrt{a};\ \ \sqrt[8]{a^3},\ \sqrt[9]{a^6},\ \sqrt[20]{a^5};\ \ \sqrt[16]{x^4},\ \sqrt[12]{x^{10}}.$

7. $\sqrt[4]{a},\ \sqrt[6]{5\,b},\ \sqrt[8]{3\,c};\ \ \sqrt[3]{2\,a},\ \sqrt[5]{3\,b},\ \sqrt[5]{4\,x}.$

8. $\sqrt{a\,x^2},\ \sqrt[39]{a^9\,x^6};\ \ \sqrt[3]{m},\ \sqrt[4]{n},\ \sqrt[6]{x},\ \sqrt[8]{m\,n\,x}.$

9. $\sqrt[10]{a^5},\ \sqrt[6]{b^3},\ \sqrt[5]{c^4};\ \ 4\sqrt[3]{5\,a^2},\ 2\sqrt[4]{2\,b^2\,x^3},\ 10\,a\sqrt{3\,c}.$

109. $\frac{2}{3}\sqrt{6} = \sqrt{(\frac{2}{3})^2 \times 6} = \sqrt{\frac{8}{3}} = \sqrt{\frac{40}{15}},$

$\frac{4}{5}\sqrt{5} = \sqrt{(\frac{4}{5})^2 \times 5} = \sqrt{\frac{16}{5}} = \sqrt{\frac{48}{15}}.\quad \therefore\ \frac{4}{5}\sqrt{5} > \frac{2}{3}\sqrt{6}.$

In general,

$$a\sqrt[n]{x} = (a^n\,x)^{\frac{1}{n}} = \sqrt[n]{a^n\,x},$$

$$b\sqrt[n]{y} = (b^n\,y)^{\frac{1}{n}} = \sqrt[n]{b^n\,y}.$$ Hence,

I. **To Compare Surds of the Same Order.** Reduce them to entire surds, and compare the resulting surd factors.

$$\tfrac{1}{2}\sqrt[3]{52} = \sqrt[3]{(\tfrac{1}{2})^3 \times 2^2 \times 13} = \sqrt[6]{\frac{13^2}{2^2}} = \sqrt[6]{42.25},$$

$$\tfrac{3}{2}\sqrt[9]{8} = \sqrt[3]{(\tfrac{3}{2})^3 \times 2} = \sqrt[6]{(\tfrac{3}{2})^6 \times 2^2} = \sqrt[6]{45.5625},$$

$$3\sqrt{\tfrac{2}{5}} = \sqrt{3^2 \times \tfrac{2}{5}} = \sqrt[6]{3^6 \times (\tfrac{2}{5})^3} = \sqrt[6]{46.656}.$$

Therefore, the order of magnitude is $3\sqrt{\frac{2}{5}},\ \frac{3}{2}\sqrt[9]{8},\ \frac{1}{2}\sqrt[3]{52}.$

In general,

$$a\sqrt[n]{x} = a^{\frac{nm}{nm}}\,x^{\frac{m}{nm}} = \sqrt[nm]{a^{nm}\,x^m},$$

$$b\sqrt[m]{y} = b^{\frac{mn}{mn}}\,y^{\frac{n}{mn}} = \sqrt[mn]{b^{mn}\,y^n}.$$ Hence,

II. **To Compare Surds of Different Orders.** Reduce them to entire surds of the same order, and compare the resulting surd factors.

Exercise 96.

Which is the greater?

1. $3\sqrt{6}$ or $2\sqrt{14}$; $6\sqrt{11}$ or $5\sqrt{15\frac{3}{5}}$; $4\sqrt{6}$ or $6\sqrt{4}$.

2. $10\sqrt{5}$ or $4\sqrt{31}$; $\frac{2}{3}\sqrt{7}$ or $\frac{3}{5}\sqrt{10}$; $\sqrt[3]{2}$ or $\sqrt[5]{3}$.

3. $\sqrt{\frac{3}{5}}$ or $\sqrt[3]{\frac{14}{15}}$; $\sqrt[4]{4}$ or $\sqrt[5]{5}$; $\sqrt{\frac{5}{3}}$ or $\sqrt[3]{1\frac{1}{2}}$.

4. $\sqrt[5]{1\frac{1}{3}}$ or $\sqrt[3]{\frac{2}{3}}$; 1.6 or $\frac{1}{2}\sqrt{10}$; $\sqrt[3]{6.5}$ or $\sqrt{3.5}$.

Arrange in order of magnitude:

5. $\sqrt{3}$, $\sqrt[3]{4}$, $\sqrt[4]{7}$; $8\sqrt{2}$, $5\sqrt{5}$, $4\sqrt{7\frac{3}{4}}$.

6. $2\sqrt[6]{21}$, $3\sqrt[3]{49}$, $4\sqrt{7}$; $3\sqrt[3]{4}$, $4\sqrt[3]{1\frac{3}{4}}$, $2\sqrt[3]{13\frac{3}{4}}$.

7. $3\sqrt[3]{2}$, $3\sqrt{2}$, $\frac{3}{2}\sqrt[4]{4}$; $2\sqrt[3]{21}$, $3\sqrt[3]{8}$, $2\sqrt{8}$.

Show that the following are similar surds:

8. $\sqrt{40}$, $\sqrt{90}$, $\sqrt{\frac{8}{5}}$; $\frac{1}{2}\sqrt{20}$, $\frac{1}{3}\sqrt{45}$, $5\sqrt{\frac{4}{5}}$.

9. $7\sqrt{\frac{9}{5}}$, $\sqrt[4]{\frac{25}{16}}$, $3\sqrt{\frac{9}{80}}$; $\sqrt[4]{162}$, $3\sqrt[4]{32}$, $\sqrt[4]{2592}$.

10. $\sqrt{27}$, $\sqrt{192}$, $\sqrt{147}$, $\sqrt{\frac{1}{3}}$; $a\sqrt[3]{m^3n^2}$, $b\sqrt[3]{8n^5}$, $\frac{a}{b}\sqrt[3]{\frac{m^6}{n}}$.

11. $3\sqrt[3]{\frac{a^3}{b}}$, $\sqrt[3]{8b^2}$, $\frac{1}{5}\sqrt[3]{\frac{a^6}{b^4}}$; $\sqrt{\frac{a^4c}{b^3}}$, $\sqrt[4]{\frac{a^4c^6}{b^2m^4}}$, $n\sqrt{\frac{a^6c^3m^6}{b^3n^6}}$.

12. $\sqrt{a^{-\frac{3}{2}}}$, $\sqrt[4]{\frac{b^4}{a^3}}$, $\sqrt[8]{\frac{a^2}{b^{8n}}}$, $\sqrt[12]{\frac{a^3x^{12m}}{b^{36n}y^{24m}}}$.

110. Addition and Subtraction of Surds.

$$\begin{aligned}
\sqrt[3]{54} &= \sqrt[3]{3^3 \times 2} = 3\sqrt[3]{2},\\
-\sqrt[3]{6\tfrac{3}{4}} &= -\sqrt[3]{\frac{3^3 \times 2}{2^2 \times 2}} = -\tfrac{3}{2}\sqrt[3]{2},\\
\tfrac{3}{4}\sqrt{\tfrac{2}{9}} &= \tfrac{3}{4}\sqrt{\frac{1}{3^2} \times 2} = \tfrac{1}{4}\sqrt{2},\\
-\tfrac{1}{2}\sqrt{\tfrac{128}{9}} &= -\tfrac{1}{2}\sqrt{\frac{8^2}{3^2} \times 2} = -\tfrac{4}{3}\sqrt{2}.
\end{aligned}$$

Adding,

$$\begin{aligned}
\sqrt[3]{54} - \sqrt[3]{6\tfrac{3}{4}} + \tfrac{3}{4}\sqrt{\tfrac{2}{9}} - \tfrac{1}{2}\sqrt{\tfrac{128}{9}} &= 3\sqrt[3]{2} - \tfrac{3}{2}\sqrt[3]{2} + \tfrac{1}{4}\sqrt{2} - \tfrac{4}{3}\sqrt{2}\\
&= \tfrac{3}{2}\sqrt[3]{2} - 1\tfrac{1}{12}\sqrt{2}.
\end{aligned}$$

$$\begin{aligned}
\sqrt[3]{\frac{27a^5x}{2b}} &= \sqrt[3]{\frac{3^3a^3 \times a^2x \times (2b)^2}{2b \times (2b)^2}} = \frac{3a}{2b}\sqrt[3]{4a^2b^2x},\\
-a\sqrt[3]{\frac{a^2x}{2b}} &= -a\sqrt[3]{\frac{a^2x \times (2b)^2}{2b \times (2b)^2}} = -\frac{a}{2b}\sqrt[3]{4a^2b^2x},\\
\frac{1}{b}\sqrt[3]{\frac{4b^2x}{a^{-2}}} &= \frac{1}{b}\sqrt[3]{\frac{4b^2x \times a^2}{a^{-2} \times a^2}} = \frac{1}{b}\sqrt[3]{4a^2b^2x}.
\end{aligned}$$

Adding,

$$\begin{aligned}
\sqrt[3]{\frac{27a^5x}{2b}} - a\sqrt[3]{\frac{a^2x}{2b}} + \frac{1}{b}\sqrt[3]{\frac{4b^2x}{a^{-2}}} &= \frac{3a}{2b}\sqrt[3]{4a^2b^2x} - \frac{a}{2b}\sqrt[3]{4a^2b^2x} + \frac{1}{b}\;\overline{4a^2b^2x}\\
&= \frac{a+1}{b}\sqrt[3]{4a^2b^2x}. \quad \text{Hence,}
\end{aligned}$$

Reduce each surd to its simplest form. Prefix the sum or difference of the rational factors to the common surd factor of the similar surds. Connect dissimilar surds by their signs.

Exercise 97.

Simplify:

1. $3\sqrt{45} - 2\sqrt{20} + 3\sqrt{5}$; $3\sqrt{\tfrac{2}{5}} + 2\sqrt{\tfrac{1}{10}}$.

2. $2\sqrt{\tfrac{3}{4}} + 3\sqrt{\tfrac{1}{3}}$; $2\sqrt[3]{162} - \tfrac{1}{2}\sqrt[3]{48}$; $3\sqrt[3]{\tfrac{1}{4}} + \sqrt[3]{\tfrac{1}{32}}$.

3. $\sqrt{3} + \sqrt{1\frac{1}{3}} - 2\sqrt{5\frac{1}{3}}$; $5\sqrt[3]{-54} - 2\sqrt[3]{-16} + 4\sqrt[3]{686}$.

4. $3\sqrt{147} - \frac{7}{3}\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{27}}$; $\sqrt{\frac{5}{3}} + 2\sqrt{60} - 3\sqrt{15} - \frac{2}{3}\sqrt{\frac{3}{5}}$.

5. $3\sqrt[4]{\frac{4}{25}} + \sqrt[6]{.001} - 4\sqrt{\frac{1}{40}}$; $3\sqrt[4]{162} - 7\sqrt[4]{32} + \sqrt[4]{1250}$.

6. $\sqrt[9]{a^6} + \frac{1}{2}\sqrt[6]{a^4} - 3\sqrt[3]{27a^2}$; $\sqrt[3]{40} - 3\sqrt[3]{320} + 4\sqrt[3]{135}$.

7. $\sqrt{50\,a\,b^3\,c^5} - \sqrt{32\,a^3\,b\,c^3} - (4\,b\,c^2 - 3\,a\,c)\sqrt{2\,a\,b\,c}$.

8. $x\sqrt[10]{m^{12}\,n^{14}\,x^6} - m\sqrt[15]{m^3\,n^{21}\,x^{24}} + n\sqrt[20]{m^{24}\,n^8\,x^{32}}$.

9. $\sqrt{3\,a\,b^2 + 6\,a\,b + 3\,a} + \sqrt{3\,a\,b^2 - 6\,a\,b + 3\,a}$.

10. $\sqrt{\dfrac{a+b}{a-b}} + \sqrt{\dfrac{a-b}{a+b}} - \dfrac{2\,a + a^2 - b^2}{a^2 - b^2}\sqrt{a^2 - b^2}$.

11. $\frac{3}{2}\sqrt[3]{\frac{2}{49}} + 0.8\sqrt{\frac{8}{3}} - \frac{1}{15}\sqrt{96} + 1.5\sqrt[3]{\frac{7}{4}} - \frac{11}{28}\sqrt[3]{1750} + 8\sqrt{\frac{3}{2}}$.

12. $\sqrt[6]{\dfrac{m^{12}\,x^3}{n^9}} + \sqrt[6]{\dfrac{m^6\,x^9}{n^3\,y^6}} - \sqrt[6]{\dfrac{m^6\,x^3\,y^6}{n^3\,z^6}}$.

111. Multiplication of Surds.

$$3\sqrt{2} \times 7\sqrt{6} = 21\sqrt{4 \times 3} = 42\sqrt{3}.$$

$$\sqrt[3]{2} \times \sqrt[4]{3} = \sqrt[12]{2^4 \times 3^3} = \sqrt[12]{432}.$$

$$\tfrac{8}{9}\sqrt{2} \times \tfrac{3}{4}\sqrt[3]{3} \times \tfrac{3}{2}\sqrt[4]{\tfrac{1}{2}} \times \sqrt[3]{\tfrac{1}{3}} = \tfrac{8}{9} \times \tfrac{3}{4} \times \tfrac{3}{2}\sqrt[12]{2^6 \times 3^4 \times (\tfrac{1}{2})^3 \times (\tfrac{1}{3})^4}$$
$$= \sqrt[12]{2^3} = \sqrt[4]{2}.$$ In general,

$$a\sqrt[n]{x} \times b\sqrt[n]{y} = a\,x^{\frac{1}{n}} \times b\,y^{\frac{1}{n}} = a\,b\,(x\,y)^{\frac{1}{n}} = a\,b\sqrt[n]{x\,y}.$$

$$a\sqrt[n]{x} \times b\sqrt[m]{y} = a\,x^{\frac{1}{n}} \times b\,y^{\frac{1}{m}} = a\,b(x^m y^n)^{\frac{1}{mn}} = a\,b\sqrt[mn]{x^m\,y^n}.$$ Hence,

I. **To Find the Product of two or more Monomials.** Reduce the surds to the same order (if necessary). Prefix the product of the rational factors to the product of the surd factors.

Multiplicand,	$3\sqrt{2}-2\sqrt[3]{5}$
Multiplier,	$3\sqrt{3}+2\sqrt[3]{6}$
$3\sqrt{2}-2\sqrt[3]{5}$ multiplied by $3\sqrt{3}$,	$9\sqrt{6}-6\sqrt[6]{5^2\times 3^3}$
$3\sqrt{2}-2\sqrt[3]{5}$ multiplied by $2\sqrt[3]{6}$,	$6\sqrt[6]{2^3\times 6^2}-4\sqrt[3]{30}$
Sum of partial products,	$9\sqrt{6}+6\sqrt[6]{288}-6\sqrt[6]{675}-4\sqrt[3]{30}.$

Hence,

II. **To Find the Product of two Polynomials.** Proceed as in Art. 24.

$$(3\sqrt{2}+2\sqrt{3})(3\sqrt{2}-2\sqrt{3})=(3\times 2^{\frac{1}{2}})^2-(2\times 3^{\frac{1}{2}})^2=3^2\times 2-2^2\times 3=6.$$

$$(a\sqrt{x}+b\sqrt{y})\,(a\sqrt{x}-b\sqrt{y})=(a\,x^{\frac{1}{2}})^2-(b\,y^{\frac{1}{2}})^2=a^2x-b^2y.$$ Hence,

III. *The product of the sum and difference of two binomial quadratic surds is a rational expression.*

Exercise 98.

Simplify:

1. $2\sqrt{3}\times 3\sqrt{3}$; $3\sqrt{\frac{2}{7}}\times\frac{7}{3}\sqrt{162}$; $\frac{7}{2}\sqrt{10}\times\frac{10}{7}\sqrt{12\frac{1}{2}}$.

2. $\frac{1}{3}\sqrt[3]{4}\times 3\sqrt[3]{2}$; $2\sqrt{14}\times\sqrt{21}$; $3\sqrt[3]{\frac{2}{3}}\times 6\sqrt{\frac{1}{2}}$.

3. $3\sqrt[5]{3}\times 3\sqrt{2}$; $(5\sqrt{3}-5)\times 2\sqrt{3}$; $\sqrt[4]{64}\times 2\sqrt{2}$.

4. $(\sqrt{2}+\sqrt{3}+2\sqrt{5})\times\sqrt{2}$; $4\sqrt[3]{75}\times 2\sqrt[3]{45}$.

5. $\frac{1}{2}\sqrt{4}\times\sqrt[4]{10}$; $\frac{1}{2}\sqrt{\frac{1}{2}}\times\frac{2}{3}\sqrt[3]{\frac{2}{3}}$; $\sqrt{5}\times\sqrt[3]{2}$.

6. $3\sqrt{\frac{2}{3}}\times\sqrt[3]{\frac{3}{2}}$; $\frac{1}{3}\sqrt{\frac{2}{3}}\times 9\sqrt[3]{\frac{3}{4}}\times\sqrt[4]{2}$.

7. $2\sqrt{3} \times \sqrt[3]{2} \times \frac{1}{3}\sqrt[3]{\frac{1}{6}}$; $\sqrt{\frac{9}{12}} \times \sqrt[4]{\frac{2}{3}}$; $\sqrt[3]{168} \times \sqrt[3]{147}$.

8. $\sqrt[3]{-9^4} \times \sqrt[3]{9} \times \sqrt[3]{9^4}$; $(3\sqrt{2}-3\sqrt{6}-\sqrt{8}+3\sqrt{20}) \times 3\sqrt{2}$.

9. $\sqrt{5} \times \sqrt[4]{10}$; $(\sqrt{n} - \sqrt{m}) \times \sqrt{n}$; $4\sqrt[3]{\frac{1}{16}} \times 3\sqrt{8}$.

10. $\sqrt{mn} \times \sqrt[3]{3m^2x} \times \sqrt{2nx}$.

11. $\sqrt[4]{m^3nx^3} \times \sqrt[5]{m^2nx}$; $2\sqrt{a} \times \sqrt[6]{b} \times 3\sqrt[4]{a} \times \sqrt[6]{b}$.

12. $\sqrt[5]{2a} \times \sqrt[3]{3a} \times \sqrt[5]{\dfrac{1}{3a}}$; $\sqrt[3]{(4mx^2)^n} \times \sqrt[3]{(2m^2x)^n}$.

13. $\dfrac{3}{n}\sqrt{\dfrac{a^2}{n}} \times \dfrac{4}{3}\sqrt{\dfrac{n^3}{2a^4}}$; $(\sqrt{2} - 3\sqrt{3})(2\sqrt{3} + 3\sqrt{2})$.

14. $(3\sqrt{5} - 4\sqrt[3]{2})(2\sqrt[3]{5} + 3\sqrt{2})$; $(\sqrt[3]{2} + \sqrt[3]{3})^2$.

15. $(5\sqrt{3} - 6\sqrt{2} + \sqrt{5})(10\sqrt{3} + 12\sqrt{2} - 2\sqrt{5})$.

16. $(\sqrt{2} + \sqrt[5]{3} + \sqrt[10]{4})(\sqrt[5]{2} - \sqrt{3})$; $\sqrt[3]{24} \times 6\sqrt[6]{3}$.

17. $(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})$; $(\sqrt[3]{3} + \sqrt[3]{4})(\sqrt[3]{3} - \sqrt[3]{4})$.

18. $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$; $(\sqrt{5} + 2\sqrt{3})(\sqrt{5} - 2\sqrt{3})$.

19. $\sqrt[3]{12 + \sqrt{19}} \times \sqrt[3]{12 - \sqrt{19}}$; $\sqrt[3]{16} \times \sqrt{8}$.

20. $\sqrt[3]{9 + \sqrt{17}} \times \sqrt[3]{9 - \sqrt{17}}$; $\sqrt{3} \times \sqrt[3]{2} \times \sqrt[3]{\frac{1}{6}}$.

21. $(\sqrt[9]{a^3} + \sqrt[12]{n^2})(\sqrt[6]{a^2} - \sqrt[18]{n^3})$; $\sqrt{\frac{2}{5}} \times \sqrt[3]{\frac{3}{8}}$.

22. $\sqrt[5]{10+\sqrt{68}} \times \sqrt[5]{10-\sqrt{68}}$; $\sqrt[3]{3\,x} \times \sqrt[5]{\dfrac{1}{3\,x^2}}$.

23. $(5\sqrt{x} + 3\sqrt{a - \frac{8}{3}x})(5\sqrt{x} - 3\sqrt{a - 2\frac{2}{3}x})$.

24. $\sqrt[8]{\frac{4}{81}} \times \sqrt[5]{\frac{27}{32}}$; $\sqrt{\frac{125}{16}} \times \sqrt[3]{\frac{4}{5}}$; $(\sqrt[4]{\frac{25}{64}})^6 (\sqrt[4]{\frac{16}{25}})^7$.

112. To Rationalize Surd Denominators of Fractions.

$$\frac{2}{5\sqrt{3}} = \frac{2 \times \sqrt{3}}{5\sqrt{3} \times \sqrt{3}} = \frac{2\sqrt{3}}{15} = \frac{2}{15} \times 1.732 + = .23 + .$$

$$\frac{2}{\sqrt[5]{9}} = \frac{2 \times \sqrt[5]{3^{5-2}}}{\sqrt[5]{3^2} \times \sqrt[5]{3^{5-2}}} = \frac{2\sqrt[5]{3^3}}{\sqrt[5]{3^5}} = \frac{2}{3}\sqrt[5]{27}.$$ In general,

$$\frac{a}{b\sqrt[n]{x^m}} = \frac{a \times \sqrt[n]{x^{n-m}}}{b\sqrt[n]{x^m} \times \sqrt[n]{x^{n-m}}} = \frac{a\sqrt[n]{x^{n-m}}}{b\sqrt[n]{x^n}}\,(n > m) = \frac{a\sqrt[n]{x^{n-m}}}{b\,x}.$$ Hence,

I. If the Fraction be of the Form $\dfrac{a}{b\sqrt[n]{x^m}}$. Multiply both terms by $\sqrt[n]{x^{n-m}}$.

$$\frac{3+\sqrt{5}}{3-\sqrt{5}} = \frac{(3+\sqrt{5}) \times (3+\sqrt{5})}{(3-\sqrt{5}) \times (3+\sqrt{5})} = \frac{3^2 + 2\,(3)\,(\sqrt{5}) + (\sqrt{5})^2}{3^2 - (\sqrt{5})^2}$$
$$= \frac{14 + 6\sqrt{5}}{9-5} = \frac{7 + 3 \times 2.236+}{2} = 6.854+.$$

$$\frac{4\sqrt{3} + 3\sqrt{5}}{2\sqrt{7} + 3\sqrt{2}} = \frac{(4\sqrt{3} + 3\sqrt{5}) \times (2\sqrt{7} - 3\sqrt{2})}{(2\sqrt{7} + 3\sqrt{2}) \times (2\sqrt{7} - 3\sqrt{2})}$$
$$= \frac{8\sqrt{21} + 6\sqrt{35} - 12\sqrt{6} - 9\sqrt{10}}{10}.$$ In general,

$$\frac{a}{\sqrt{b} \pm \sqrt{c}} = \frac{a \times (\sqrt{b} \mp \sqrt{c})}{(\sqrt{b} \pm \sqrt{c}) \times (\sqrt{b} \mp \sqrt{c})} = \frac{a(\sqrt{b} \mp \sqrt{c})}{(b^{\frac{1}{2}})^2 - (c^{\frac{1}{2}})^2} = \frac{a(\sqrt{b} \mp \sqrt{c})}{b - c}.$$

$$\frac{a}{b \pm \sqrt{c}} = \frac{a \times (b \mp \sqrt{c})}{(b \pm \sqrt{c}) \times (b \mp \sqrt{c})} = \frac{a(b \mp \sqrt{c})}{(b)^2 - (c^{\frac{1}{2}})^2} = \frac{a(b \mp \sqrt{c})}{b^2 - c}.$$ Hence,

II. **If the Denominator is a Binomial Involving only Quadratic Surds.** Multiply both terms of the fraction by the terms of the denominator with a different sign between them.

Note. It is often useful to change a fraction which has a surd in its denominator to an equivalent one with a surd in its numerator. Thus,

$$\frac{3}{\sqrt{5}} = \frac{3 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{3\sqrt{5}}{5} = \tfrac{3}{5} \times 2.236+ = 1.3416+.$$

Exercise 99.

Rationalize the denominators of:

1. $\dfrac{2}{\sqrt{2}+\sqrt{3}}$; $\dfrac{3}{2\sqrt{5}-\sqrt{6}}$; $\dfrac{2-\sqrt{2}}{1+\sqrt{2}}$; $\dfrac{3\sqrt{5}}{\sqrt{3}+\sqrt{2}}$.

2. $\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}}$; $\dfrac{2\sqrt{5}-\sqrt{2}}{\sqrt{5}+3\sqrt{2}}$; $\dfrac{1}{3-2\sqrt{6}}$; $\dfrac{6}{\sqrt[5]{64}}$.

3. $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$; $\dfrac{3x-\sqrt{xy}}{\sqrt{xy}-3y}$; $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$.

4. $\dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$; $\dfrac{a}{\sqrt{a+\sqrt{b}}}$; $\dfrac{1}{\sqrt{5}-\sqrt[4]{2}}$; $\dfrac{2a}{3\sqrt[r]{x^m z^n}}$.

Given $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$; find the approximate values of:

5. $\dfrac{20}{\sqrt{2}}$; $\sqrt{50}$; $8\frac{1}{3}\sqrt{288}$; $\dfrac{25}{\sqrt{5}}$; $\dfrac{9}{2\sqrt{675}}$; $\dfrac{5}{\sqrt{500}}$.

6. $\dfrac{1+\sqrt{2}}{2+\sqrt{2}}$; $\dfrac{1-\sqrt{5}}{3+\sqrt{5}}$; $\dfrac{3}{2\sqrt{2}-3\sqrt{3}}$; $\dfrac{1}{\sqrt{5}-\sqrt{2}}$; $\dfrac{1}{2+\sqrt{3}}$.

113. Division of Surds.

$$2\sqrt{54} \div 3\sqrt{6} = \tfrac{2}{3}\sqrt{\tfrac{54}{6}} = \tfrac{2}{3} \times 3 = 2.$$

$$\tfrac{3}{5}\sqrt[6]{72} \div \tfrac{5}{6}\sqrt{2} = \frac{18}{25}\sqrt[6]{\frac{72}{2^3}} = \tfrac{18}{25}\sqrt[6]{3^2} = \tfrac{18}{25}\sqrt[3]{3}.$$

$$\sqrt{\tfrac{3}{4}} \div \sqrt{\tfrac{2}{3}} = \sqrt{\tfrac{3}{4} \div \tfrac{2}{3}} = \sqrt{\tfrac{9}{8}} = \tfrac{3}{4}\sqrt{2}.$$

$$\sqrt{\tfrac{2}{3}} \div \sqrt[3]{\tfrac{1}{3}} = \sqrt[6]{(\tfrac{2}{3})^3 \div (\tfrac{1}{3})^2} = \sqrt[6]{\tfrac{8}{3}} = \tfrac{1}{3}\sqrt[6]{1944}.$$

In general, $a\sqrt[n]{x} \div b\sqrt[n]{y} = \frac{a}{b}\left(\frac{x}{y}\right)^{\frac{1}{n}} = \frac{a}{b}\sqrt[n]{\frac{x}{y}}.$

$$a\sqrt[n]{x} \div b\sqrt[m]{y} = \frac{a}{b}\left(\frac{x^m}{y^n}\right)^{\frac{1}{mn}} = \frac{a}{b}\sqrt[mn]{\frac{x^m}{y^n}}.$$ Hence,

I. **If the Divisor is a Monomial.** Reduce the surds to the same order (if necessary). Prefix the quotient of the rational factors to the quotient of the surd factors.

$$2\sqrt{3} \div (3\sqrt{3} + 2\sqrt{2}) = \frac{3\sqrt{3}}{3\sqrt{3}+2\sqrt{2}} = \frac{3\sqrt{3}\times(3\sqrt{3}-2\sqrt{2})}{(3\sqrt{3}+2\sqrt{2})\times(3\sqrt{3}-2\sqrt{2})}$$

$$= \frac{27-6\sqrt{6}}{19}.$$ Hence, in general,

II. **If the Divisor is a Binomial Involving only Quadratic Surds.** Express the quotient in the form of a fraction, and rationalize its denominator.

$$\sqrt{a}+\sqrt{b}-\sqrt{c}\,)\,a + 2\sqrt{ab} + b - c\,(\,\sqrt{a}+\sqrt{b}+\sqrt{c}.$$

Divisor multiplied by $\sqrt{a}$, $\quad a + \sqrt{ab} - \sqrt{ac}$

First remainder, $\quad \sqrt{ab} + b + \sqrt{ac} - c$

Divisor multiplied by $\sqrt{b}$, $\quad \sqrt{ab} + b - \sqrt{bc}$

Second remainder, $\quad \sqrt{ac} + \sqrt{bc} - c$

Divisor multiplied by $\sqrt{c}$, $\quad \sqrt{ac} + \sqrt{bc} - c$

Hence, in general,

III. **To Divide a Polynomial by a Polynomial.** Proceed as in Art. 33.

Exercise 100.

Simplify:

1. $21\sqrt{384} \div 8\sqrt{98}$; $5\sqrt{27} \div 3\sqrt{24}$; $\sqrt{12} \div \sqrt[3]{24}$.

2. $-13\sqrt{125} \div 5\sqrt{65}$; $6\sqrt{14} \div 2\sqrt{21}$.

3. $\dfrac{3\sqrt{11}}{2\sqrt{98}} \div \dfrac{5}{7\sqrt{22}}$; $\dfrac{3\sqrt{48}}{5\sqrt{112}} \div \dfrac{6\sqrt{84}}{\sqrt{394}}$; $\sqrt{\dfrac{3}{2}} \div \sqrt[3]{\dfrac{2}{3}}$.

4. $1\frac{1}{3}\sqrt[3]{2\frac{2}{3}} \div \frac{2}{3}\sqrt{1\frac{1}{3}}$; $\sqrt[5]{12} \div \sqrt[3]{2}$; $\sqrt{6} \div \sqrt[8]{4}$.

5. $20\sqrt[6]{200} \div 4\sqrt{2}$; $\sqrt[3]{18} \div \sqrt{6}$; $4\sqrt[6]{32} \div \sqrt[3]{16}$.

6. $\dfrac{\sqrt{40}}{3\sqrt{108}} \times \dfrac{7\sqrt{12}}{5\sqrt{14}} \div \dfrac{2\sqrt{60}}{15\sqrt{84}}$; $(\sqrt[3]{8^2} + \sqrt{4^3}) \div \sqrt[4]{16^3}$.

7. $(15\sqrt{105} - 36\sqrt[4]{100} + 30\sqrt[8]{81}) \div 3\sqrt{15}$.

8. $\sqrt[6]{0.064} \div \sqrt{10}$; $a\sqrt[3]{a\,b^2 c} \div \sqrt[3]{a\,b}$; $\sqrt{a} \div \sqrt[3]{a\,b}$.

9. $\dfrac{3}{m-n}\sqrt{\dfrac{2x}{m-n}} \div \sqrt{\dfrac{18x^3}{(m-n)^5}}$; $4\sqrt{\dfrac{2}{3}} \div 3\sqrt[3]{\dfrac{3}{2}}$.

10. $(a\,c\,x^2\sqrt{y} - b\,c\,y\sqrt{x}) \div c\sqrt{x\,y}$; $\sqrt{a\,x} \div \sqrt[3]{a^2 x^2}$.

11. $\sqrt[3]{4\,m\,n^2} \div \sqrt{2\,m^3 n}$; $\sqrt[4]{2\,m^3 n^2} \times \sqrt[3]{m^5 n^3} \div \sqrt{m^3 n^5}$.

12. $\sqrt[4]{4\,m^2 n^2} \times \sqrt[6]{9\,m^2 n^4} \div \sqrt[12]{25\,m^2 n^6}$; $\sqrt[3]{3\,a^2 x} \div \sqrt[5]{\dfrac{a\,x^2}{2}}$.

13. $\dfrac{a}{b}\sqrt{\dfrac{m\,y}{x}} \times \dfrac{m}{n}\sqrt[3]{\dfrac{n\,x}{y^2}} \div \dfrac{a\,m}{bn}\sqrt[4]{\dfrac{y^2}{m^2 n^2}}$; $(x-1) \div (\sqrt{x}-1)$.

14. $\sqrt[8]{10.4976} \div 2\sqrt{6}$; $(2x - \sqrt{xy}) \div (2\sqrt{xy} - y)$.

15. $(3\sqrt{3} + 2\sqrt{2}) \div (\sqrt{3} + \sqrt{2})$; $4\sqrt[3]{ax} \div 3\sqrt{ax}$.

16. $\dfrac{2\sqrt{15}+8}{5+\sqrt{15}} \div \dfrac{8\sqrt{3}-6\sqrt{5}}{5\sqrt{3}-3\sqrt{5}}$; $\dfrac{8-4\sqrt{5}}{1+\sqrt{5}} \div \dfrac{3\sqrt{5}-7}{5+\sqrt{7}}$.

17. $(\sqrt[9]{x^6} + \sqrt[6]{x^2y^3} + \sqrt[2n]{y^{2n}}) \div (\sqrt[6n]{x^{2n}} - \sqrt[12]{x^2y^3} + \sqrt[6]{y^3})$.

114. Involution and Evolution of Surds.

$$\left[\frac{1}{3}\sqrt{\frac{2}{5}}\right]^3 = \left[\frac{1}{3} \times \left(\frac{2}{5}\right)^{\frac{1}{2}}\right]^3 = \frac{1}{3^3} \times \left(\frac{2}{5}\right)^{\frac{3}{2}} = \frac{2}{135}\sqrt{\frac{2}{5}} = \frac{2}{675}\sqrt{10}.$$

$$\sqrt[5]{486a\sqrt[3]{4a^2}} = [3^5 \times 2a(2^2a^2)^{\frac{1}{3}}]^{\frac{1}{5}} = [3^5 \times 2^{\frac{5}{3}}a^{\frac{5}{3}}]^{\frac{1}{5}} = 3 \times 2^{\frac{1}{3}}a^{\frac{1}{3}} = 3\sqrt{2a}.$$

In general, $\quad (a^{m_1}\sqrt[n]{b^m})^p = \left(a^{m_1}b^{\frac{m}{n}}\right)^p = a^{m_1p}b^{\frac{mp}{n}} = a^{m_1p}\sqrt[n]{b^{mp}}.$

$$\sqrt[r]{a^{m_1}\sqrt[n]{b^m}} = \left(a^{m_1}b^{\frac{m}{n}}\right)^{\frac{1}{r}} = a^{\frac{m_1}{r}}b^{\frac{m}{nr}}. \quad \text{Hence,}$$

Express the surd factors with fractional exponents, and proceed as in Art. 104.

EXAMPLE 1.

$$\left(\frac{a\sqrt{a}}{\sqrt[6]{c^5}} - \frac{\sqrt[4]{c}}{2a}\right)^3 = \left[\frac{a^{\frac{3}{2}}}{c^{\frac{5}{6}}} - \frac{c^{\frac{1}{4}}}{2a}\right]^3$$

$$= \left(\frac{a^{\frac{3}{2}}}{c^{\frac{5}{6}}}\right)^3 - 3\left(\frac{a^{\frac{3}{2}}}{c^{\frac{5}{6}}}\right)^2\left(\frac{c^{\frac{1}{4}}}{2a}\right) + 3\left(\frac{a^{\frac{3}{2}}}{c^{\frac{5}{6}}}\right)\left(\frac{c^{\frac{1}{4}}}{2a}\right)^2 - \left(\frac{c^{\frac{1}{4}}}{2a}\right)^3$$

$$= \frac{a^{\frac{9}{2}}}{c^{\frac{5}{2}}} - 3\left(\frac{a^3}{c^{\frac{5}{3}}}\right)\left(\frac{c^{\frac{1}{4}}}{2a}\right) + 3\left(\frac{a^{\frac{3}{2}}}{c^{\frac{5}{6}}}\right)\left(\frac{c^{\frac{1}{2}}}{2^2a^2}\right) - \frac{c^{\frac{3}{4}}}{2^3a^3}$$

$$= \frac{a^{\frac{9}{2}}}{c^{\frac{5}{2}}} - \frac{3a^2}{2c^{\frac{17}{12}}} + \frac{3}{4a^{\frac{1}{2}}c^{\frac{1}{3}}} - \frac{c^{\frac{3}{4}}}{8a^3}$$

$$= \frac{a^4\sqrt{a}}{c^2\sqrt{c}} - \frac{3a^2}{2c\sqrt[12]{c^5}} + \frac{3}{4\sqrt[6]{a^3c^2}} - \frac{\sqrt[4]{c^3}}{8a^3}.$$

EXAMPLE 2. Find the square root of $4a - 12\sqrt[6]{a^3b^2} + 9\sqrt[3]{b^2} + 16\sqrt[4]{a^2c} - 24\sqrt[12]{b^4c^3} + 16\sqrt{c}$.

Process. $4a + 16a^{\frac{1}{2}}c^{\frac{1}{4}} - 12a^{\frac{1}{2}}b^{\frac{1}{3}} + 16c^{\frac{1}{2}} - 24b^{\frac{1}{3}}c^{\frac{1}{4}} + 9b^{\frac{2}{3}} \;\Big|\; 2a^{\frac{1}{2}} + 4c^{\frac{1}{4}} - 3b^{\frac{1}{3}}$

$= 2\sqrt{a} + 4\sqrt[4]{c} - 3\sqrt[3]{b}$.

First term of root squared, $4a$

First remainder, $16a^{\frac{1}{2}}c^{\frac{1}{4}} - 12a^{\frac{1}{2}}b^{\frac{1}{3}} + 16c^{\frac{1}{2}} - 24b^{\frac{1}{3}}c^{\frac{1}{4}} + 9b^{\frac{2}{3}}$

First trial divisor, $4a^{\frac{1}{2}}$

First complete divisor, $4a^{\frac{1}{2}} + 4c^{\frac{1}{4}}$

First complete divisor multiplied by $4c^{\frac{1}{4}}$, $16a^{\frac{1}{2}}c^{\frac{1}{4}} \qquad + 16c^{\frac{1}{2}}$

Second remainder, $-12a^{\frac{1}{2}}b^{\frac{1}{3}} \qquad - 24b^{\frac{1}{3}}c^{\frac{1}{4}} + 9b^{\frac{2}{3}}$

Second trial divisor, $4a^{\frac{1}{2}} + 8c^{\frac{1}{4}}$

Second complete divisor, $4a^{\frac{1}{2}} + 8c^{\frac{1}{4}} - 3b^{\frac{1}{3}}$

Second complete divisor multiplied by $-3b^{\frac{1}{3}}$, $-12a^{\frac{1}{2}}b^{\frac{1}{3}} \qquad - 24b^{\frac{1}{3}}c^{\frac{1}{4}} + 9b^{\frac{2}{3}}$

Exercise 101.

Find the values of the following:

1. $(\sqrt[5]{3})^3$; $\sqrt[4]{\sqrt{81}}$; $(\sqrt[6]{12})^3$; $\sqrt[3]{\sqrt[4]{512}}$; $(\sqrt[12]{32})^3$.

2. $\sqrt[-4]{\sqrt[3]{256}}$; $\sqrt[4]{\frac{4}{9}\sqrt[3]{\frac{4}{9}}}$; $(\sqrt[8]{x-y})^4$; $\sqrt[2]{\sqrt[3]{64}}$.

3. $(\sqrt[8]{64})^3$; $\sqrt{n\sqrt[3]{n^{-\frac{1}{2}}}}$; $(\sqrt[3]{4})^2$; $\sqrt[5]{\sqrt{32}}$; $(\sqrt[9]{8})^4$.

4. $\sqrt[5]{x^2\sqrt{x}}$; $(\sqrt[9]{27})^4$; $\sqrt[-3]{\frac{8}{27}}$; $(2\sqrt[6]{3a^3})^2$; $\sqrt[-2]{a\sqrt[3]{a}}$.

5. $(2\sqrt[6]{a^4 b})^3$; $\sqrt[n]{\sqrt[x]{\left(\frac{1}{a}\right)^m}}$; $[\sqrt[mn]{x}]^n$; $(2a\sqrt[3]{2bc})^4$.

6. $\sqrt[-x]{3^{-x}a^{-x}}$; $(3\sqrt[3]{24a^4x^6})^3$; $\sqrt[\frac{2}{3}]{4\frac{1}{3}a^2}$; $\sqrt[-\frac{3}{2}]{27x^{-6}}$.

7. $\sqrt[-4]{n^{-3}\sqrt[5]{n^{-5}}}$; $\left[\sqrt[x]{(a-c)^{\frac{1}{x}}}\right]^n$; $\sqrt[-\frac{6}{5}]{1}$; $[\frac{1}{2}\sqrt[3]{4a^2c}]^5$.

8. $\sqrt[3]{\frac{a}{3}\sqrt{\frac{a}{3}}}$; $\left(\frac{1}{a^n}\sqrt{a}\right)^{-3}$; $\sqrt[-4]{a^{-3}\sqrt[4]{a^{-3}}}$; $(a\sqrt[3]{a})^{-3}$.

9. $\left(n^x\sqrt[x]{n^{\frac{1}{x}}}\right)^{-x}$; $\sqrt[\frac{n}{m}]{a^{-x}b^{-c}\sqrt[cx]{a^x b^c}}$; $(\sqrt[7]{m^3n})^3(\sqrt[7]{m^3n^{12}})^4$.

10. $\sqrt[-n]{x^n\sqrt[\frac{1}{n^2}]{x^{\frac{1}{n}}}}$; $\sqrt[4]{\left(\frac{a\sqrt{c}}{\sqrt[3]{ac}}\right)^3}$; $\sqrt[-\frac{5}{4}]{\frac{32a^{-10}x^{-\frac{15}{2}}}{m^{\frac{3}{4}}n^{\frac{5}{8}}}}$.

11. $\sqrt[\frac{2a}{3}]{9^{-a}x^{-a}}$; $[(x+y)\sqrt{xy}]^2$; $\sqrt[7]{128\sqrt[5]{243a^{70}}}$.

Find the values of the following, and express the results in terms of positive exponents, by inspection:

12. $(\sqrt[4]{x^2+y^2})^8$; $(\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}})^2$; $(\sqrt{2}+\sqrt{3})^3$.

13. $(\sqrt[3]{3}-1)^3$; $(\sqrt{2}+\sqrt[3]{4})^3$; $\left[\dfrac{n\sqrt{n}}{\sqrt[4]{m^6}}-\dfrac{\sqrt[3]{m}}{n}\right]^3$.

14. $\left[\sqrt[n]{\left(a^{-\frac{n}{2}}-a^{\frac{n}{2}}\right)^4}\right]^n$; $\left[\left(\sqrt[m]{a^n}\right)^p-\left(\sqrt[\frac{1}{m}]{b^{-\frac{1}{n}}}\right)^{1}\right]^3$; $(2^{-1}-1)^4$.

15. $\left[\left(a^{\frac{4}{3}}\right)^{-3}+a^0\right]^5$; $\left[\left(\sqrt[-n]{a^{-n}}\right)^p+\left(\sqrt[\frac{1}{m}]{b^{-\frac{1}{n}}}\right)^{-\frac{n}{m}}\right]^4$.

16. $\left[\dfrac{n}{2m}\sqrt{\dfrac{2m}{n}}+\dfrac{2\sqrt[3]{m}}{\sqrt[3]{n}}\right]^5$; $\left[2\sqrt{\dfrac{m^{-1}}{n^4}}-\dfrac{4\sqrt[3]{x^{3a}}}{\sqrt[3]{m^{-3}}}\right]^3$.

Find the square roots of:

17. $\dfrac{x^{-2}}{16}+1+9\sqrt[3]{y^2}+\frac{1}{2}x^{-1}-\frac{3}{2}x^{-1}\sqrt[3]{y}-6\sqrt[3]{y}$.

18. $1-2^{n+1}+4^n$; $9^n-2^{n+1}\times 3^n+4^n$.

19. $\dfrac{y^2}{x}+\dfrac{x^2}{4y}+2y\sqrt[4]{\dfrac{y}{x}}-x\sqrt[4]{\dfrac{x}{y}}$; $a^{2\sqrt{2}}-2+a^{-2\sqrt{2}}$.

20. $\sqrt[3n]{x^{2m}}-4\sqrt[6n]{x^{5m}}+4x^{\frac{m}{n}}+2\sqrt[6n]{x^{7m}}-4\sqrt[3n]{x^{4m}}+\sqrt[3n]{x^{5m}}$.

Miscellaneous Exercise 102.

Find the values of the following:

1. $\sqrt[\frac{9}{2}]{27}$; $\sqrt[-2]{\frac{25}{36}}$; $\sqrt{3}\times\sqrt{27}$; 1^{-n}; $\sqrt[-\frac{2}{3}]{4}$; $\sqrt{\frac{7}{12}}\div\sqrt{\frac{3}{7}}$.

2. $\sqrt[3]{\sqrt[5]{32a^{45}}}$; $\sqrt[n]{6\sqrt[5]{m^{5n}}}$; $\sqrt[-5]{\dfrac{13\frac{1}{2}}{11\frac{1}{2}}}\div\sqrt[-5]{\dfrac{16\frac{1}{5}}{18\frac{2}{5}}}$; $\sqrt[3\frac{1}{2}]{128}$.

Reduce the following to their simplest forms:

3. $\dfrac{4}{5}\sqrt[3]{\dfrac{8x^6}{25a^2}}$; $\sqrt{3\sqrt[3]{3}}$; $\sqrt[4]{3888}$; $\dfrac{m+n}{n}\sqrt{\dfrac{mn^2}{16(m+n)}}$.

4. $\sqrt[3a]{\frac{x}{y}}$; $(x-y)\sqrt[n]{\frac{x+y}{2^n(x-y)^n}}$; $\frac{a}{b}\sqrt[n+1]{a\,b^{3n+3}}$.

Reduce to equivalent surds of the same lowest order:

5. $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{5}$; $\sqrt[3]{9}$, $\sqrt[4]{8}$, $\sqrt{5}$; $3\sqrt{75}$, $3\sqrt[3]{54}$.

6. $2\sqrt{18}$, $3\sqrt[3]{16}$, $4\sqrt[4]{162}$, $5\sqrt[6]{128}$; $\sqrt[7]{a^2}$, $\sqrt[4]{b^3}$, $\sqrt[14]{x^{14n}}$.

Change to similar surds:

7. $\sqrt[6]{27}$, $\sqrt[4]{144}$; $\sqrt[3]{54}$, $3\sqrt[9]{8}$; $\sqrt[3]{72}$, $\sqrt[3]{243}$.

8. $\sqrt{0.2}$, $\sqrt{5}$; $\sqrt[8]{25}$, $\sqrt[4]{405}$; $\frac{1}{2}\sqrt[6]{\frac{8}{125}}$, $\sqrt{\frac{2}{45}}$.

9. $2\sqrt[3]{\frac{a^3b^2}{3}}$, $\sqrt[3]{72\,b^5}$, $\frac{1}{2}\sqrt[3]{\frac{243\,a^6}{b^4}}$; $\frac{5}{6}\sqrt[6]{.16}$, $\sqrt[\frac{3}{2}]{2.5}$.

10. $\sqrt[3]{\frac{8}{3}}$, $\sqrt[6]{\frac{81}{64}}$, $\sqrt[3]{\frac{72}{343}}$, $3\sqrt[12]{\frac{1}{81}}$; $\sqrt{20}$, $3\sqrt[4]{\frac{1}{25}}$, $4\sqrt{125}$.

11. $\sqrt[10]{32}$, $\sqrt[12]{128}$; $\sqrt[4]{64\,x^{12}}$, $\sqrt{\frac{32\,b^8}{9}}$, $\sqrt[6]{\sqrt[x]{(8\,a^6)^x}}$.

12. $\sqrt[10]{\frac{9}{256}}$, $\sqrt[5]{192}$, $\sqrt[5]{\frac{3}{512}}$, $\sqrt[5]{\frac{6}{3125}}$, $\sqrt[5]{\frac{2}{81}}$; $\sqrt[\frac{3}{4}]{2}$, $\sqrt[\frac{1}{3}]{2\frac{1}{9}}$.

Arrange in order of magnitude:

13. $3\sqrt{2\frac{1}{18}}$, $4\sqrt[3]{4\frac{1}{4}}$, $3\sqrt[3]{3}$; $5\sqrt[9]{8}$, $3\sqrt[6]{9}$, $3\sqrt{19}$.

14. $\frac{5}{3}\sqrt[3]{2}$, $\frac{2}{3}\sqrt{3}$, $\frac{1}{3}\sqrt{5}$, $2\sqrt[3]{8}$; $\sqrt{\frac{4}{5}}$, $\sqrt[3]{\frac{11}{15}}$.

15. $a^{\frac{n}{3}}\sqrt[3]{(8\,a\,b^2)^n}$, $\frac{2^n}{\sqrt[3]{b}}\sqrt[3]{(2^3\,a^2\,b^3)^n}$, $(64)^n\sqrt{\frac{a^{4n}\,b^{4n}}{(256)^n}}$.

Find the values of:

16. $\sqrt{243}+\sqrt{27}+\sqrt{48}$; $2\sqrt[3]{189}+3\sqrt[3]{875}-7\sqrt[3]{56}$.

17. $4\sqrt{5}\times\sqrt[3]{11}$; $3\sqrt[6]{500}\div\sqrt[2]{5}$; $\sqrt{2\sqrt[3]{2\sqrt{2}}}\div\sqrt[3]{\sqrt{2\sqrt[3]{2}}}$.

18. $5\sqrt{2}+3\sqrt{8}-2\sqrt{32}$; $3\sqrt[3]{81}-4\sqrt[3]{192}+\sqrt[3]{648}$.

19. $\sqrt[3]{128}\times\sqrt[3]{432}$; $\frac{1}{5}\sqrt{5}\times\frac{1}{2}\sqrt[3]{2}\times\sqrt[6]{80}\times\sqrt[3]{5}$.

20. $\sqrt[9]{64}+5\sqrt[3]{32}-\sqrt[3]{108}$; $\frac{1}{6}(\sqrt[9]{27}+\frac{1}{2}\sqrt[3]{192}+\sqrt[3]{81})$.

21. $2\sqrt[6]{\frac{125}{27}}+\sqrt{60}-\sqrt[4]{225}-\sqrt{\frac{3}{5}}$; $\frac{1}{2}\sqrt{\frac{1}{2}}\div(\sqrt{2}+3\sqrt{\frac{1}{2}})$.

22. $(\sqrt[5]{\frac{4}{5}}\div 2\sqrt[10]{\frac{1}{400}})\div\sqrt[5]{16}$; $(\sqrt[3]{9}-2\sqrt[3]{2\frac{2}{3}}+4\sqrt[3]{\frac{1}{3}})2\sqrt[6]{9}$.

23. $(6\sqrt[6]{\frac{4}{9}}+\sqrt[3]{18})\div\sqrt[3]{72}$; $1\frac{3}{5}\sqrt{20}-3\sqrt{5}-\sqrt{\frac{1}{5}}$.

24. $\frac{12}{5}\sqrt[3]{60}\div(2\sqrt[3]{240}+7\sqrt[3]{3\frac{3}{4}})$; $\sqrt{\frac{3}{16}}\div\left(\frac{3}{ab}\right)^{\frac{1}{2}}$.

25. $(\sqrt[3]{16}-2\sqrt[6]{4}+4\sqrt[3]{54})(5\sqrt[9]{64}+3\sqrt[12]{\frac{1}{16}}-2\sqrt[15]{32})$.

Rationalize the denominators of:

26. $\dfrac{\sqrt{20}-\sqrt{8}}{\sqrt{5}+\sqrt{2}}$; $\dfrac{(3+\sqrt{3})(3+\sqrt{5})(\sqrt{5}-2)}{(5-\sqrt{5})(1+\sqrt{3})}$.

27. $\dfrac{m+an-\sqrt{m^2+a^2n^2}}{m-an+\sqrt{m^2+a^2n^2}}$; $\dfrac{x^{\frac{1}{m}}}{y^{\frac{1}{n}}}$; $\dfrac{2}{\sqrt{5}+\sqrt{3}-\sqrt{2}}$.

Simplify the following:

28. $\dfrac{\sqrt[3]{5}\times\sqrt[4]{3}}{\sqrt{2}}$; $\sqrt{(\frac{25}{16})^7}\times\sqrt{(\frac{64}{25})^8}$; $\sqrt[4]{(8a^3b)^2}\times\sqrt[4]{(2ab^3)^2}$.

29. $\dfrac{7+3\sqrt{5}}{7-3\sqrt{5}}+\dfrac{7-3\sqrt{5}}{7+3\sqrt{5}}$; $\sqrt[n]{\dfrac{\sqrt[4]{9^{4n+1}}\times\sqrt{3\times 3^n}}{3\sqrt{3^{-n}}}}$.

30. $(a\div\sqrt[n]{a})^n+\sqrt[1-n]{\left(a\sqrt[n]{a^{-\frac{1}{n}}}\right)^{n^2}}$; $\sqrt[n]{x^2y^{mn}}\times\sqrt[\frac{1}{mn}]{x^{\frac{1}{m}}y^2}$.

31. $\sqrt[3]{a} \times \sqrt[4]{a^{-3}} \times \sqrt[3]{a^4} \times \sqrt[12]{a} \times \sqrt[8]{a^{\frac{25}{3}}} \times \sqrt[6]{a^{-\frac{7}{4}}}$.

32. $\sqrt[-\frac{2}{3}]{\dfrac{a^{-3}}{c^{\,-\frac{3}{2}}\sqrt{b}}} \div \left(\dfrac{\sqrt[-4]{a} \times \sqrt[6]{b^3}}{a^2 c^{-1}}\right)^{-2}$; $\left[\sqrt[5]{\dfrac{a^{\frac{1}{2}}}{a^{-2}}\left(\dfrac{a}{\sqrt{a}}\right)^{\frac{1}{3}}}\right]^{\frac{1}{4}}$.

33. $\dfrac{b}{\sqrt{a}} \times \sqrt[3]{a\,c} \times \dfrac{\sqrt[4]{c^3}}{\sqrt{b}} \times \dfrac{\sqrt[-2]{b}}{\sqrt[-6]{a}}$; $\sqrt[6]{a^3 b^2}\left(\dfrac{\sqrt[4]{b}}{\sqrt[6]{a}}\right)^2 \div \sqrt[4]{\dfrac{b^{-1}}{a}}$.

34. $\sqrt[m+n]{\left(a^{1+\frac{n}{m}}\right)^m} \div \sqrt[m]{\dfrac{a^{2m}}{(a^{-1})^{-m}}}$; $\dfrac{\left(3^{3\frac{1}{2}}\right)^{\sqrt{3}}}{\left(9^{\sqrt{\frac{2}{3}}}\right)^{6\frac{1}{2}}}$; $\frac{4}{5}\left(\frac{2}{5}\right)^{\frac{2}{3}}$.

35. $\dfrac{\sqrt{x}}{\sqrt[-3]{y}}\left(\dfrac{\sqrt[4]{y}}{\sqrt[6]{x}}\right) \div \sqrt[-4]{\dfrac{y}{x^{-1}}}$; $\frac{2}{3}\sqrt[3]{\frac{3}{2}} + \frac{1}{4}\sqrt[3]{\frac{256}{9}}$; $-2\left(\frac{2}{3}\right)^{\frac{2}{3}}$.

36. $\sqrt[3]{27^2} + \sqrt[4]{16^3} - \dfrac{2}{\sqrt[3]{8^{-2}}} + \dfrac{\sqrt[5]{2}}{\sqrt[-5]{4^2}}$; $\dfrac{a}{b}\left(\dfrac{c^2\,m^2}{a^2\,n^2}\right)^{\frac{1}{2}}$.

37. $\dfrac{\sqrt{a^3} + a\,b}{a\,b - b^3} - \dfrac{\sqrt{a}}{\sqrt{a} - b}$; $\dfrac{\sqrt[3]{x^2} - 4\sqrt[3]{x^{-2}}}{\sqrt[3]{x^2} + 4 + 4\sqrt[-3]{x^2}}$.

38. $\dfrac{x - 7\sqrt{x}}{x - 5\sqrt{x} - 14} \div \left(1 + \dfrac{2}{\sqrt{x}}\right)^{-1}$; $(\sqrt[3]{a^2} + \sqrt[6]{a})^3$.

39. $\left[\sqrt[3]{\left(\frac{125}{64}\right)^5} \div \sqrt[4]{\left(\frac{625}{32}\right)^{10}}\right] \times \sqrt[\frac{1}{a}]{(5^2 n^2 m^3)^{\frac{5}{a}}} \times \sqrt[\frac{x}{3}]{(10^{10} n^{20} m^{30})^{-\frac{x}{6}}}$.

40. Express $\dfrac{\sqrt[3]{5} \times \sqrt[4]{3}}{\sqrt[5]{2}}$ with a single radical sign.

Queries. What sign is given to the nth power? To the nth root? Why? How change the order of a surd? In I., Art. 112, why take m less than n? How rationalize a surd denominator? What powers of negative numbers are positive? What negative?

Imaginary Expressions.

115. An **Imaginary Expression** is an indicated *even* root of a *negative* expression; as, $\sqrt[4]{-a}$; $a+b\sqrt{-1}$. $\sqrt{-1}$ is an imaginary *square root;* $a\sqrt[4]{-1}$ is an imaginary *fourth root;* etc.

$$\sqrt{-a^2}=\sqrt{a^2\times(-1)}=\sqrt{a^2}\times\sqrt{-1}=a\sqrt{-1}.$$

$$\sqrt{-b}\ =\sqrt{b\times(-1)}=\sqrt{b}\times\sqrt{-1}.\quad \text{Hence,}$$

Every imaginary square root can be expressed as the product of a rational or surd factor multiplied by $\sqrt{-1}$.

The successive powers of $\sqrt{-1}$ are found as follows:

$$[\sqrt{-1}]^1=[(-1)^{\frac{1}{2}}]^1=(-1)^{\frac{1}{2}}=+\sqrt{-1};$$
$$[\sqrt{-1}]^2=[(-1)^{\frac{1}{2}}]^2=(-1)\ =-1;$$
$$[\sqrt{-1}]^3=[(-1)^{\frac{1}{2}}]^3=(-1)^{\frac{3}{2}}=(-1)\ (-1)^{\frac{1}{2}}=-\sqrt{-1};$$
$$[\sqrt{-1}]^4=[(-1)^{\frac{1}{2}}]^4=(-1)^2=+1;$$
$$[\sqrt{-1}]^5=[(-1)^{\frac{1}{2}}]^5=(-1)^{\frac{5}{2}}=(-1)^2(-1)^{\frac{1}{2}}=+\sqrt{-1};$$
$$[\sqrt{-1}]^6=[(-1)^{\frac{1}{2}}]^6=(-1)^3=-1;$$
$$[\sqrt{-1}]^7=[(-1)^{\frac{1}{2}}]^7=(-1)^{\frac{7}{2}}=(-1)^3(-1)^{\frac{1}{2}}=-\sqrt{-1};$$
$$[\sqrt{-1}]^8=[(-1)^{\frac{1}{2}}]^8=(-1)^4=+1;$$
$$[\sqrt{-1}]^9=[(-1)^{\frac{1}{2}}]^9=(-1)^{\frac{9}{2}}=(-1)^4(-1)^{\frac{1}{2}}=+\sqrt{-1};$$ and so on. In general,

$$[\sqrt{-1}]^n=[(-1)^{\frac{1}{2}}]^n=(-1)^{\frac{n}{2}}=\sqrt{(-1)^n}\quad=\pm\sqrt{-1}\text{ or }\mp 1,$$

according as n is an *odd* or *even* integer. Hence,

The successive powers of $\sqrt{-1}$ form the repeating series:

$$+\sqrt{-1},\ -1,\ -\sqrt{-1},\ +1.$$

The methods for operating with imaginary expressions are the same as those for surds; but before applying the methods it is better to remove the factor $\sqrt{-1}$. All cases of multiplication can be made a direct application of Arts. 97, 114.

Illustrations. $\sqrt{-8a^2b^3} = \sqrt{8a^2b^3 \times (-1)} = \sqrt{8a^2b^3} \times \sqrt{-1}$
$= 2ab\sqrt{2b} \times \sqrt{-1}.$

$$\sqrt{-4x^2} = 2x\sqrt{-1}.$$

$$\begin{aligned}\sqrt{-9a^2} + \sqrt{-49a^2} - \sqrt{4a^2} &= 3a\sqrt{-1} + 7a\sqrt{-1} - 2a \\ &= 10a\sqrt{-1} - 2a \\ &= 2a(5\sqrt{-1} - 1).\end{aligned}$$

$$\begin{aligned}3\sqrt{-3} \times 4\sqrt{-2} &= (3\sqrt{3} \times \sqrt{-1})(4\sqrt{2} \times \sqrt{-1}) \\ &= 3\sqrt{3} \times 4\sqrt{2} \times \sqrt{-1} \times \sqrt{-1} \\ &= 12\sqrt{3 \times 2} \times [(-1)^{\frac{1}{2}}]^2 \\ &= -12\sqrt{6}.\end{aligned}$$

$$\begin{aligned}&2\sqrt{-3} \times 5\sqrt{-2} \times 3\sqrt{-6} \\ &= 2\sqrt{3} \times 5\sqrt{2} \times 3\sqrt{6} \times \sqrt{-1} \times \sqrt{-1} \times \sqrt{-1} \\ &= 30\sqrt{3 \times 2 \times 6} \times [(-1)^{\frac{1}{2}}]^3 \\ &= -180\sqrt{-1}.\end{aligned}$$

$$\begin{aligned}-6\sqrt{-3} \div -2\sqrt{-4} &= \frac{-6\sqrt{3} \times \sqrt{-1}}{-2\sqrt{4} \times \sqrt{-1}} = \frac{-6\sqrt{3} \times \sqrt{-1}}{-4 \times \sqrt{-1}} \\ &= \tfrac{6}{4}\sqrt{3} \times 1 = \tfrac{3}{2}\sqrt{3}.\end{aligned}$$

$$\begin{aligned}(1+\sqrt{-1}) \div (1-\sqrt{-1}) &= \frac{1+\sqrt{-1}}{1-\sqrt{-1}} = \frac{(1+\sqrt{-1}) \times (1+\sqrt{-1})}{(1-\sqrt{-1}) \times (1+\sqrt{-1})} \\ &= \frac{(1+\sqrt{-1})^2}{1^2 - [\sqrt{-1}]^2} = \frac{1 + 2\sqrt{-1} + (-1)}{1 - (-1)} \\ &= \sqrt{-1}.\end{aligned}$$

EXAMPLE. Multiply $1 - 2\sqrt{-1}$ by $3 + \sqrt{-1}$.

Process.	
	$1 - 2\sqrt{-1}$
	$3 + \sqrt{-1}$
$1 - 2\sqrt{-1}$ multiplied by 3,	$3 - 6\sqrt{-1}$
$1 - 2\sqrt{-1}$ multiplied by $\sqrt{-1}$,	$2 + \sqrt{-1}$
Sum of the partial products,	$5 - 5\sqrt{-1}$

Notes: 1. Imaginary expressions represent impossible operations; yet it is a mistake to suppose that they are unreal, or that they have no importance.

2. If the student employ the method of multiplying or dividing the expressions under the radicals (Arts. 111, 113), for all cases in multiplication and division, he cannot readily determine the *sign* of the product or dividend. Thus,

$$\sqrt{-a} \times \sqrt{-a} = \sqrt{-a \times -a} = \sqrt{a^2} = \pm a.$$

3. Is the above product both $\pm a$ or $-a$? We are limited to the consideration of the product of two equal factors, and we know that the sign of each is negative; also, that $\sqrt{a^2} = \pm a$. Hence, the sign of $\sqrt{a^2}$ will necessarily be the same as that of each of these factors. Therefore, it will be the same as was its root. Thus,

$$\sqrt{-3} \times \sqrt{-3} = -\sqrt{9} = -3,$$
$$\sqrt{-2} \times \sqrt{-3} = -\sqrt{6}.$$

Exercise 103.

Simplify:

1. $\sqrt{-9}$; $\sqrt[4]{-16}$; $\sqrt{-12a}$; $\sqrt{-4a^2}$; $\sqrt{-\frac{1}{4}}$.

2. $\sqrt{-49a^{2n}b^6}$; $\sqrt[6]{-729}$; $\sqrt[10]{-a^{20}}$; $\sqrt[2n]{-2^{2n}}$.

Find the values of:

3. $(\sqrt{-1})^{10}$; $(\sqrt{-1})^{51}$; $(\sqrt{-1})^{27}$; $(-\sqrt{-1})^8$.

4. $(-\sqrt{-1})^3$; $(-\sqrt{-1})^4$; $(-\sqrt{-1})^5$; $(-\sqrt{-1})^9$.

5. $\sqrt{-25} - \sqrt{-49} + \sqrt{-121} - \sqrt{-64} + \sqrt{-1} - \sqrt{-36}$.

6. $2\sqrt{-24} + \dfrac{1}{\sqrt{-3}} - \sqrt{-18}$; $\dfrac{\sqrt{-22}}{\sqrt{-33}} - \dfrac{\sqrt{-216}}{\sqrt{-324}}$.

7. $\sqrt{-36a^6} + \sqrt{-9a^6} - \sqrt{-(1-a)^2a^6}$; $-\sqrt{-a^4}$.

8. $\sqrt{-(a-b)^2} + \sqrt{-(a^2-2ab+b^2)} + \sqrt{-16a^4b^4} - \sqrt{-4a^2}$.

Multiply:

9. $\sqrt{-3}$ by $\sqrt{-6}$; $3\sqrt{-4} + \sqrt{-9}$ by $4\sqrt{-3}$.

10. $2\sqrt{-9}$ by $4\sqrt{-4}$; $1+\sqrt{-1}$ by $1+\sqrt{-1}$.

11. $\sqrt{-2}+3\sqrt{-3}$ by $\sqrt{-2}+3\sqrt{-3}$.

12. $3-2\sqrt{-4}$ by $5+3\sqrt{-4}$; $4+\sqrt{-5}$ by $4-\sqrt{-5}$.

13. $1+\sqrt{-1}$ by $1-\sqrt{-1}$; $2-\sqrt{-3}$ by $1-2\sqrt{-3}$

14. $2\sqrt{-3}-6\sqrt{-6}$ by $\sqrt{-3}+\sqrt{-6}$.

15. $\sqrt{a-b}$ by $\sqrt{b-a}$; $a+\sqrt{-x}$ by $a-\sqrt{-x}$.

16. $a\sqrt{-a}+b\sqrt{-b}$ by $a\sqrt{-a}-b\sqrt{-b}$.

Divide:

17. $\sqrt{-9}$ by $\sqrt{-3}$; $-\sqrt{-1}$ by $-6\sqrt{-3}$.

18. $\sqrt{-5}$ by $\sqrt{-20}$; $\sqrt{-24}-\sqrt{-9}$ by $\sqrt{-3}$.

19. $2\sqrt{-4a^2}$ by $\sqrt{-a^2}$; $a+\sqrt{-a}$ by $\sqrt{-a^2}$.

20. $-2\sqrt{-1}$ by $1-\sqrt{-1}$; 2 by $1+\sqrt{-1}$.

21. $\sqrt[4]{-21}$ by $\sqrt[4]{-5}$; $\sqrt[6]{-81}$ by $\sqrt[6]{-3}$.

22. $4+\sqrt{-2}$ by $2-\sqrt{-2}$; $\sqrt{-3}$ by $1-\sqrt{-1}$.

Rationalize the denominators of:

23. $\dfrac{3+\sqrt{-3}}{2-\sqrt{-2}}$; $\dfrac{2\sqrt{-1}-3\sqrt{-2}}{4\sqrt{-2}+5\sqrt{-3}}$; $\dfrac{3+3\sqrt{-1}}{2-2\sqrt{-1}}$.

24. $\dfrac{3-\sqrt{-1}}{2+\sqrt{-1}}$; $\dfrac{a-\sqrt{-x}}{a+\sqrt{-x}}$; $\dfrac{\sqrt{-a}-\sqrt{-x}}{\sqrt{-a}+\sqrt{-x}}$.

Queries. To what form can all imaginary monomials be reduced? In multiplication and division why separate the imaginary expressions into their surd and imaginary factors? Is it necessary in all cases?

Quadratic Surds.

116. I. *A quadratic surd cannot equal the sum or difference of a rational expression and a quadratic surd.*

Proof. If possible, let $\sqrt{a} = b \pm \sqrt{c}$, in which $\sqrt{a}$ and $\sqrt{c}$ are dissimilar quadratic surds, and b a rational expression.

Square both members, $a = b^2 \pm 2\,b\sqrt{c} + c$.

Transpose, $\pm 2\,b\sqrt{c} = a - b^2 - c. \quad \therefore \sqrt{c} = \dfrac{\pm a \mp b^2 \mp c}{2\,b}.$

That is, a surd equal to a rational expression, which is impossible. Therefore, $\sqrt{a}$ cannot equal $b \pm \sqrt{c}$.

II. *If* $a + \sqrt{b} = x + \sqrt{y}$, *in which* a *and* x *are rational and* $\sqrt{b}$ *and* $\sqrt{y}$ *are quadratic surds, prove that* $a = x$ *and* $b = y$.

Proof. Transposing, $\sqrt{b} = (x - a) + \sqrt{y}$. Now if a and x were unequal, we would have a quadratic surd equal to the sum of a rational expression and a quadratic surd, which, by I., is impossible. Hence, $a = x$. Therefore, $\sqrt{b} = \sqrt{y}$, or $b = y$.

III. *If* $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, *prove that* $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

Proof. Square both members, $a + \sqrt{b} = x + 2\sqrt{x\,y} + y$.

Therefore II., $a = x + y \quad (1) \qquad$ and $\sqrt{b} = 2\sqrt{x\,y} \quad (2)$

Subtract (2) from (1), $\quad a - \sqrt{b} = x - 2\sqrt{x\,y} + y$.

Extract the square root, $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

Similarly it may be shown that if $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$,

then $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$.

Square Root of a Quadratic Surd.

117. **To find the square root of a binomial surd** $a \pm \sqrt{b}$.

Process. Let $$\sqrt{a \pm \sqrt{b}} = \sqrt{x} \pm \sqrt{y} \quad (1)$$

Then (III., Art. 116), $$\sqrt{a \mp \sqrt{b}} = \sqrt{x} \mp \sqrt{y} \quad (2)$$

Multiply (1) and (2) together, $$\sqrt{a^2 - b} = x - y \quad (3)$$

Square (1), $$a \pm \sqrt{b} = x \pm 2\sqrt{xy} + y.$$

Therefore (II., Art. 116), $$a = x + y \quad (4)$$

Add (3) and (4), $a + \sqrt{a^2 - b} = 2x$. $\therefore\ x = \dfrac{a + \sqrt{a^2 - b}}{2}$.

Subtract (3) from (4), $a - \sqrt{a^2 - b} = 2y$. $\therefore\ y = \dfrac{a - \sqrt{a^2 - b}}{2}$.

Therefore, $$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}} \quad (i)$$

Notes: 1. Evidently, unless $a^2 - b$ be a perfect square, the values of $\sqrt{x}$ and $\sqrt{y}$ will be complex surds; and the expression $\sqrt{x} + \sqrt{y}$ will not be as simple as $\sqrt{a + \sqrt{b}}$.

2. Since, $\sqrt{a^2 c} + \sqrt{bc} = \sqrt{c}(a + \sqrt{b})$, also if $a^2 - b$ be a perfect square the square root of $a + \sqrt{b}$ may be expressed in the form $\sqrt{x} + \sqrt{y}$, the square root of $\sqrt{a^2 c} + \sqrt{bc}$ is of the form $\sqrt[4]{c}(\sqrt{x} + \sqrt{y})$.

3. Frequently the square root of a binomial surd may be found by inspection. Thus,

Find two numbers whose sum is the rational term, and whose product is the square of half the radical term. Connect the square roots of these numbers by the sign of the radical term.

EXAMPLES: 1. Find the square root of $3\frac{1}{2} - \sqrt{10}$.

Process. Let $$\sqrt{x} - \sqrt{y} = \sqrt{3\tfrac{1}{2} - \sqrt{10}} \quad (1)$$

Then (III., Art. 116), $$\sqrt{x} + \sqrt{y} = \sqrt{3\tfrac{1}{2} + \sqrt{10}} \quad (2)$$

Multiply (1) and (2) together, $$x - y = \sqrt{\tfrac{49}{4} - 10} = \tfrac{3}{2} \quad (3)$$

Square (1), $$x - 2\sqrt{xy} + y = 3\tfrac{1}{2} - \sqrt{10}.$$

Therefore (II., Art. 116), $$x + y = 3\tfrac{1}{2} \quad (4)$$

From (3) and (4), $$x = 2\tfrac{1}{2},\ y = 1.$$

Therefore, $$\sqrt{3\tfrac{1}{2} - \sqrt{10}} = \sqrt{\tfrac{5}{2}} - 1 = \tfrac{1}{2}\sqrt{10} - 1.$$

We may employ the general form (i). Thus (since $a = 83$ and $\sqrt{b} = +12\sqrt{35}$),

$$2.\quad \sqrt{83 + 12\sqrt{35}} = \sqrt{\frac{83 + \sqrt{83^2 - (12\sqrt{35})^2}}{2}}$$

$$+\sqrt{\frac{83 - \sqrt{83^2 - (12\sqrt{35})^2}}{2}} = \sqrt{\frac{83+43}{2}} + \sqrt{\frac{83-43}{2}}$$

$$= \sqrt{63} + \sqrt{20} = 3\sqrt{7} + 2\sqrt{5}.$$

$$3.\quad \sqrt{\sqrt{27} - 2\sqrt{6}} = \sqrt{\sqrt{3}\,(3 - 2\sqrt{2})} = \sqrt[4]{3} \times \sqrt{3 - 2\sqrt{2}},$$

also $\sqrt{3 - 2\sqrt{2}}$ (in which $a = 3$ and $\sqrt{b} = -2\sqrt{2}$)

$$= \sqrt{\frac{3 + \sqrt{3^2 - (2\sqrt{2})^2}}{2}} - \sqrt{\frac{3 - \sqrt{3^2 - (2\sqrt{2})^2}}{2}} = \sqrt{2} - 1.$$

$$\therefore \sqrt{\sqrt{27} - 2\sqrt{6}} = \sqrt[4]{3}\,(\sqrt{2} - 1).$$

4. Find by inspection the square root of $103 - 12\sqrt{11}$.

Solution. The two numbers whose sum is 103 and whose product is $\left(\frac{12\sqrt{11}}{2}\right)^2$, are 99 and 4. Hence, $\sqrt{103 - 12\sqrt{11}}$ $= \sqrt{99} - \sqrt{4} = 3\sqrt{11} - 2$.

5. Similarly, $\sqrt{10 + 2\sqrt{21}} = \sqrt{7} + \sqrt{3}$, because 7 and 3 are the only numbers whose sum is 10 and whose product is $(\sqrt{21})^2$.

Exercise 104.

Find the square roots of:

1. $7 - 2\sqrt{10}$; $5 + 2\sqrt{6}$; $41 - 24\sqrt{2}$; $2\frac{1}{4} + \sqrt{5}$.

2. $18 - 8\sqrt{5}$; $11 + 2\sqrt{30}$; $13 - 2\sqrt{42}$.

3. $15 - \sqrt{56}$; $47 - 4\sqrt{33}$; $6 - 2\sqrt{5}$; $10 + 4\sqrt{6}$.

4. $4\frac{1}{3} - \frac{4}{3}\sqrt{3}$; $\sqrt{32} - \sqrt{24}$; $3\sqrt{5} + \sqrt{40}$.

5. $\sqrt{27} + \sqrt{15}$; $2m + 1 + 2\sqrt{m^2 + n - 2}$.

6. $(m^2 + m)n - 2mn\sqrt{m}$; $9 - 2\sqrt{14}$.

7. $(m + n)^2 - 4(m - n)\sqrt{mn}$; $3x - 2x\sqrt{2}$.

Find the fourth roots of:

8. $97 - 56\sqrt{3}$; $\frac{3}{2}\sqrt{5} + 3\frac{1}{2}$; $56 + 24\sqrt{5}$.

9. $17 + 12\sqrt{2}$; $4(31 - 8\sqrt{15})$; $248 + 32\sqrt{60}$.

Simple Equations Containing Surds.

118. EXAMPLES: 1. Solve $\sqrt{4x^2 - 7x + 1} = 2x - 1\frac{4}{5}$ (1)

Process. Square (1), $4x^2 - 7x + 1 = 4x^2 - 7\frac{1}{5}x + \frac{81}{25}$.
$\therefore\ x = 11\frac{1}{5}$. Hence,

To Solve an Equation containing a Single Surd. Arrange the terms so as to have the surd alone in one member, and then raise each member to the power indicated by the root index.

Note. If the equation contains *two* or *more surds*, two or more operations may be necessary in order to clear it of radicals. Thus,

2. Solve $\sqrt{25x - 29} - \sqrt{4x - 11} = 3\sqrt{x}$.

Process. Transpose, $\sqrt{25x - 29} = 3\sqrt{x} + \sqrt{4x - 11}$ (1)
Square (1), $25x - 29 = 9x + 6\sqrt{(4x - 11)x} + 4x - 11$.
Transpose, etc., $\sqrt{(4x - 11)x} = 2x - 3$ (2)
Square (2), $4x^2 - 11x = 4x^2 - 12x + 9$. $\therefore\ x = 9$.

3. Solve $\dfrac{\sqrt{x}+2m}{\sqrt{x}+n}=\dfrac{\sqrt{x}+4m}{\sqrt{x}+3n}$ (1)

Process. Clear (1) of fractions, transpose and unite, etc.,

$(m-n)\sqrt{x}=mn.$ Hence, $\sqrt{x}=\dfrac{mn}{m-n}.$ $\therefore\ x=\left(\dfrac{mn}{m-n}\right)^2.$

4. Solve $\dfrac{\sqrt{m+x}+\sqrt{m-x}}{\sqrt{m+x}-\sqrt{m-x}}=n.$

Process. Rationalize the denominator,

$$\frac{m+\sqrt{m^2-x^2}}{x}=n \tag{1}$$

From (1), $\sqrt{m^2-x^2}=nx-m$ (2)

Square (2), $m^2-x^2=n^2x^2-2mnx+m^2.$

Transpose, etc., $x^2(1+n^2)=2mnx.$

Divide by x, $x(1+n^2)=2mn.$ $\therefore\ x=\dfrac{2mn}{1+n^2}.$

Exercise 105.

Solve:

1. $\sqrt{x+5}=4$; $\sqrt{3x+6}=6$; $\sqrt{x^2-2}=x+2.$

2. $\sqrt{x^2-3x+5}=x-1$; $\sqrt[3]{2x-3}-1=2.$

3. $\sqrt{3+\sqrt{4+\sqrt{x-4}}}=2$; $\sqrt{4+x\sqrt{x^2+32}}=x+2.$

4. $\sqrt[n]{mx+a}=\sqrt[n]{cx+b}$; $\sqrt{x^2+\sqrt{4-\sqrt{3x}}}=x.$

5. $\sqrt{x+2}=2\sqrt{2x-3}$; $\sqrt{3x+5}=3\sqrt{2x+1}.$

6. $\sqrt{3x}-4=\sqrt{2x+16}$; $\sqrt[4]{2x-4}=\sqrt{4-\sqrt{2x}}.$

7. $3\sqrt{x}=\dfrac{8}{\sqrt{9x-32}}+\sqrt{9x-32}.$

8. $\sqrt{x+3}+\sqrt{x+8}-\sqrt{4x+21}=0.$

9. $\sqrt{\sqrt{x}+5}-\sqrt{\sqrt{x}-5}=\sqrt{2\sqrt{x}}.$

10. $\sqrt[4]{m^2+x\sqrt{n^2+x^2}}=\sqrt{x+m};\ \ \sqrt{x}+\sqrt{x-2}=\dfrac{3}{\sqrt{x}}.$

11. $\sqrt[4]{4+2\sqrt{2x-5}}=\sqrt{3};\quad \dfrac{\sqrt{x}+\sqrt{m}}{\sqrt{x}-\sqrt{m}}=\dfrac{m}{n}.$

12. $\dfrac{\sqrt{2x+1}+3\sqrt{x}}{\sqrt{2x+1}-3\sqrt{x}}=3;\ \ \dfrac{\sqrt{mx}-n}{\sqrt{mx}+n}=\dfrac{3\sqrt{mx}-2n}{3\sqrt{mx}+5n}.$

13. $\dfrac{\sqrt{5x}+\sqrt{5}}{\sqrt{3x}+\sqrt{3}}=\dfrac{\sqrt{x}+5}{\sqrt{x}+3};\ \ \dfrac{\sqrt{6x}+2}{\sqrt{6x}-2}=\dfrac{4\sqrt{6x}+6}{4\sqrt{6x}-9}.$

14. $\sqrt{\dfrac{n}{m+x}}+\sqrt{\dfrac{a}{m-x}}=\sqrt[4]{\dfrac{4an}{m^2-x^2}}.$

Solve the following for x and y:

15. $x+4\sqrt{3}+y=15-x+y\sqrt{5}.$

16. $x+y+x\sqrt{a}+y\sqrt{b}=1-\sqrt{a}.$

17. $x-5+(2y-3)\sqrt{3}=5x-\sqrt{12}.$

18. $x-a+(y-3)\sqrt{a}+b=nx+\sqrt{ab}.$

19. $a-\sqrt{x+y}=y-x-\sqrt{m+n}.$

20. $x\sqrt{m}(\sqrt{m}+1)=n-m+y\sqrt{n}(1-\sqrt{n}).$

CHAPTER XX.

LOGARITHMS.

119. If $a^l = m$, then l is called *the logarithm of* m *to the base* a. Hence,

A **Logarithm** is the exponent by which a certain number, called the **base**, must be affected in order to produce a given number.

The logarithm of m to the base a is written $\log_a m$. Thus, $\log_a m = l$ expresses the relation $a^l = m$; $\log_{10} 100 = 2$ expresses the relation $10^2 = 100$, etc.

Since numbers are formed by combinations of tens, any number may be expressed, exactly or approximately, as a power of 10. Thus,

$$2912 = 10^2 \times 20 + 10^2 \times 9 + 10 + 2; \text{ or } 2912 = 10^{3.464\cdots};$$

$$1000 = 10^3; \text{ etc.}$$

120. **Common System of Logarithms**. This system has 10 for its base, and is the only one used for practical calculations. Thus,

Since $10^0 = 1$, $\log 1 = 0$; since $10^1 = 10$, $\log 10 = 1$;
since $10^2 = 100$, $\log 100 = 2$; since $10^3 = 1000$, $\log 1000 = 3$;
since $10^4 = 10000$, $\log 10000 = 4$; and so on.

Since $10^{-1} = \frac{1}{10} = .1$, $\log .1 = -1 = 9 - 10$;
since $10^{-2} = \frac{1}{100} = .01$, $\log .01 = -2 = 8 - 10$;
since $10^{-3} = \frac{1}{1000} = .001$, $\log .001 = -3 = 7 - 10$; and so on.

It is evident that the logarithm of all numbers greater than 1 is *positive*, and of all numbers between 0 and 1 is *negative;* also, that the logarithm of any numbers between

1 and 10	is	$0 +$ a fraction;
10 and 100	is	$1 +$ a fraction;
100 and 1000	is	$2 +$ a fraction;
1 and .1	is	$-1 +$ a fraction, or $9 +$ a fraction -10;
.1 and .01	is	$-2 +$ a fraction, or $8 +$ a fraction -10;
.01 and .001	is	$-3 +$ a fraction, or $7 +$ a fraction -10; and so on.

It thus appears that the logarithm of a number consists of an integral part, called the **characteristic**, and a fractional part, called the **mantissa**.

The mantissa is always made positive.

Illustrations. It is known that $\log 5 = 0.69897$; $\log 12 = 1.07918$; $\log 2912 = 3.46419$; etc. These results mean that $10^{0.69897} = 5$; $10^{1.07918} = 12$; $10^{3.46419} = 2912$; etc.

Notes: 1. The fractional part of a logarithm cannot be expressed exactly, but an approximate value may be found, true to as many decimal places as desired. Thus, the logarithm of 3 is found to be 0.477121, true to the sixth place.

2. For brevity the expression "logarithm of 3" is written log 3. The expression "log x" is read "logarithm of x."

3. Logarithms were invented by John Napier, Baron of Merchiston, Scotland, and first published in 1614.

4. There are only two systems of logarithms in general use: the *Natural*, or *Hyperbolic*, system, and the *Briggsian*, or *Common*, system. The base subscript of the former is e, and that of the latter is 10.

5. The natural system, invented by John Speidell and published in 1619, is employed in the higher branches of analysis and in scientific investigations; its base is 2.718281828+.

6. The common system, more properly called the **denary** or *decimal* system, was invented by Henry Briggs, an English geometrician, and first published in 1617. The logarithm of its base, 10, is *always* 1.

7. The logarithms invented by Napier are entirely different from those invented by Speidell, though they are closely connected with them. The natural system may be regarded as a modification of the *original Napierian* system.

121. Since $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$, $\log 1000 = 3$, etc., the *characteristic* of the logarithms of all numbers consisting of *one*

integral digit (that is, all numbers with one figure to the left of its decimal point) is 0; of all numbers consisting of *two* integral digits is 1; of all numbers consisting of *three* integral digits is 2; and so on. Hence,

I. *The characteristic of the logarithm of an integral number, or of a mixed decimal, is one less than the number of integral places.*

Since $\log .1 = -1$, $\log .01 = -2$, $\log .001 = -3$, etc.; the *characteristic* of the logarithm of any decimal whose *first* significant figure occupies the *first* decimal place (that is, of any number between 0.1 and 1) is -1; of any decimal whose *first* significant figure occupies the *second* decimal place (that is, of any number between 0.01 and 0.1) is -2; of any decimal whose *first* significant figure occupies the *third* decimal place (that is, of any number between 0.001 and 0.01) is -3; and so on. Hence,

II. *The characteristic of the logarithm of a decimal is negative, and is numerically equal to the number of the place occupied by the first significant figure of the decimal.*

The *characteristic only* is negative. Hence, in the case of decimals whose logarithms are *negative*, the logarithm is made to consist of a *negative* characteristic and a *positive* mantissa. To indicate this, the minus sign is written *over* the characteristic, or else 10 is added to the characteristic and the subtraction of 10 from the logarithm is indicated.

Thus, $\log .0012 = \bar{3}.0792$, or $7.0792 - 10$; read "characteristic minus three, mantissa nought seven ninety-two," or "characteristic *seven minus ten*, etc." In reading the mantissa, for brevity, two integers are read at a time. Thus, $\log 2 = 0.30103$, is read "the logarithm of two equals characteristic zero, mantissa thirty ten three."

Illustrations. The characteristic of the logarithm of 9 is 0; of 32 is 1; of 433 is 2; of 39562 is 4; of 632.526 is 2; of .42 is -1; of .023622 is -2; of .0000325 is -5; etc.

122. Let m and n be any two numbers whose logarithms are x and y in the common system. Then $10^x = m$ and $10^y = n$. Multiplying the equations together, we have $10^{x+y} = mn$. Hence (Art. 119), $\log mn = x + y$. But $x = \log m$ and $y = \log n$. Therefore, $\log mn = \log m + \log n$. Similarly $\log mnp = \log m + \log n + \log p$; etc. Hence,

The logarithm of a product is found by adding together the logarithms of its factors.

Illustrations. Given $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 5 = 0.6990$, $\log 7 = 0.8451$.

$$\begin{aligned}\log 252 &= \log(2 \times 2 \times 3 \times 3 \times 7)\\ &= \log 2 + \log 2 + \log 3 + \log 3 + \log 7\\ &= 2 \log 2 + 2 \log 3 + \log 7\\ &= 2 \times 0.3010 + 2 \times 0.4771 + 0.8451\\ &= 0.6020 + 0.9542 + 0.8451\\ &= 2.4013.\end{aligned}$$

$$\begin{aligned}\log 300 &= \log(2 \times 3 \times 5 \times 10)\\ &= \log 2 + \log 3 + \log 5 + \log 10\\ &= 0.3010 + 0.4771 + 0.6990 + 1\\ &= 2.4771.\end{aligned}$$

Exercise 106.

Given $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 5 = 0.6990$, $\log 7 = 0.8451$; find the values of the following:

1. log 6; log 64; log 14; log 8; log 12; log 15; log 84.

2. log 343; log 16; log 216; log 27; log 45; log 36.

3. log 90; log 210; log 3600; log 1120; log 1680.

123. If any number be multiplied or divided by any integral power of 10, since the *sequence* of the digits in the resulting number remains the *same*, the mantissæ of their logarithms will be unaffected. Thus, since it is known that $\log 577.932 = 2.7619$,

$$\begin{aligned}
\log 5779.32 &= \log(577.932 \times 10) &&= \log 577.932 + \log 10 \\
&&&= \quad 2.7619 + 1 \qquad = 3.7619. \\
\log 57793.2 &= \log(577.932 \times 100) &&= \log 577.932 + \log 100 \\
&&&= \quad 2.7619 + 2 \qquad = 4.7619. \\
\log 57.7932 &= \log(577.932 \times 0.1) &&= \log 577.932 + \log 0.1 \\
&&&= \quad 2.7619 + (-1) = 1.7619. \\
\log 5.77932 &= \log(577.932 \times 0.01) &&= \log 577.932 + \log 0.01 \\
&&&= \quad 2.7619 + (-2) = 0.7619. \\
\log .577932 &= \log(577.932 \times 0.001) &&= \log 577.932 + \log 0.001 \\
&&&= \quad 2.7619 + (-3) = \bar{1}.7619.
\end{aligned}$$

Etc. Hence,

The mantissæ of the logarithms of numbers having the same sequence of digits are the same.

Illustrations. If $\log 44.068 = 1.6441$, $\log 4.4068 = 0.6441$, $\log .44068 = \bar{1}.6441$ or $9.6441 - 10$, $\log .000044068 = \bar{5}.6441$ or $5.6441 - 10$, $\log 440.68 = 2.6441$, $\log 4406800 = 6.6441$, etc. If $\log 2 = 0.3010$, $\log .2 = \bar{1}.3010$, $\log .02 = \bar{2}.3010$, $\log 20 = 1.3010$, etc. Hence,

The **mantissa** *depends only on the* **sequence of digits**, *and the* **characteristic** *on the* **position of the decimal point**.

Exercise 107.

1. Write the characteristics of the logarithms of: 12753; 13.2; 532; .053; .2; .37; .00578; .000000735; 1.23041.

2. The mantissa of log 6732 is .8281, write the logarithms of: 6.732; 673.2; 67.32; .6732; .006732; .000006732.

3. Name the number of digits in the integral part of the numbers whose logarithms are: 5.3010; 0.6990; 3.4771.

4. Name the place occupied by the first significant figure in the numbers whose logarithms are: $\bar{4}.8451$; 0.7782.

Given $\log 2 = 0.3010$, $\log 3 = 0.4771$; $\log 5 = 0.6990$, $\log 7 = 0.8451$; find the logarithms of the following numbers:

5. .18; 22.5; 1.05; 3.75; 10.5; 6.3; .0125; 420.

6. .0056; .128; 14.4; 1.25; 12.5; .05; .0000315.

7. .3024; 5.4; .006; .0021; 3.5; .00035; 4.48.

124. Let m be any number whose logarithm is x. Then $10^x = m$. Raising both members to the pth power, we have $10^{px} = m^p$. Hence (Art. 119), $\log m^p = px$. But $x = \log m$. Therefore, $\log m^r = p \log m$. Similarly $\log m^p n^q = p \log m + q \log n$, etc. Hence,

The logarithm of any power of a number is found by multiplying the logarithm of the number by the exponent of the power.

Illustrations. $\log 5^{10} = 10 \log 5 = 10 \times 0.6990 = 6.9900$.
$\log .003^5 = 5 \log .003 = 5 \times \bar{3}.4771 = \overline{13}.3855$.
$\log 864 = \log 2^5 \times 3^3 = 5 \log 2 + 3 \log 3 = 5 \times 0.3010 + 3 \times 0.4771$
$= 1.5050$.

Note. If the number is a decimal and the exponent positive, the product of the characteristic and exponent will be negative, and since the mantissa is *made* positive, we must algebraically add whatever is carried from the mantissa.

Thus, $\log .0005^{10} = 10 \times \bar{4}.6990 = \overline{40} + 6.9900 = \overline{34} + 0.9900 = \overline{34}.9900$.

Exercise 108.

Given $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 5 = 0.6990$, $\log 7 = 0.8451$; find the logarithms of:

1. 2^4; 5^3; 7^3; 8^3; 3^7; 64; 81; 72; $(8.1)^7$; $(2.10)^5$.

2. 343; .036; .000128; $(.0336)^{10}$; $(.06174)^2$; $(3.84)^9$.

125. Let m and n be any two numbers whose logarithms are x and y. Then $10^x = m$ and $10^y = n$. $\therefore\ 10^{x-y} = m \div n$. $\log \frac{m}{n} = x - y = \log m - \log n$. Similarly, $\log \frac{m\,n}{m_1\,n_1} = \log m + \log n - (\log m_1 + \log n_1)$. Etc. Hence,

The logarithm of a quotient is found by subtracting the logarithm of the divisor from the logarithm of the dividend.

Illustrations. $\log \frac{3}{2} = \log 3 - \log 2 = 0.4771 - 0.3010 = 0.1761$.
$\log \frac{5}{7} = \log 5 - \log 7 = (0.6990) - 0.8451 = (1.6990 - 1) - 0.8451$
$= 0.8539 - 1 = \bar{1}.8539$.

Note. To subtract a greater logarithm from a less logarithm. Add to the characteristic of the minuend the least number which will make the minuend greater than the subtrahend; also indicate the subtraction of the same number from the minuend so increased. Then proceed as before. Thus,

$$\log \frac{252}{300} = \log 252 - \log 300 = (2.4014) - 2.4771 = (3.4014 - 1) - 2.4771 = \bar{1}.9243.$$

$$\log \frac{.005}{.07} = (\bar{3}.6990) - \bar{2}.8451 = (\bar{1}.6990 - 2) - \bar{2}.8451 = 0.8539 - 2 = \bar{2}.8539 \text{ or}$$
$8.8539 - 10$.

Exercise 109.

Given $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 5 = 0.6990$, $\log 7 = 0.8451$; find the logarithms of:

1. $\frac{5}{2}$; $\frac{3}{.05}$; $\frac{7}{5}$; $\frac{3}{5}$; $3\frac{1}{2}$; $\frac{.003}{.07}$; $\frac{.005}{.02}$; $\left(\frac{5}{10}\right)^5$; $\frac{.007}{.02}$.

2. $4\frac{2}{3}$; $\frac{.7^3}{.005^2}$; 1.25; $\frac{.03}{5}$; $\frac{315}{8.1}$; $\frac{.343}{.000027}$; 5; $\frac{.02}{.007}$.

126. Let m be any number of which the logarithm is x. Then $10^x = m$. Taking the rth root of each member, we have $10^{\frac{x}{r}} = \sqrt[r]{m}$. $\therefore$ (Art. 119), $\log \sqrt[r]{m} = \frac{x}{r} = \frac{\log x}{r}$. Similarly, $\log \sqrt[r]{m\,n} = \frac{\log m + \log n}{r}$. Etc. Hence,

The logarithm of any root of a number is found by dividing the logarithm of the number by the index of the root.

Illustrations. $\log \sqrt[5]{5} = \frac{\log 5}{5} = \frac{0.6990}{5} = 0.1398.$

$$\log \sqrt[7]{.0007} = \frac{\log .0007}{7} = \frac{\bar{4}.8451}{7} = \frac{3.8451 + \bar{7}}{7} = 0.5493 + \bar{1} = \bar{1}.5493.$$

$$\log \sqrt[6]{1.5^5} = \frac{\log 1.5^5}{6} = \frac{5 \log 3 \times .5}{6} = \frac{5\,(\log 3 + \log .5)}{6}$$

$$= \frac{5\,(0.4771 + \bar{1}.6990)}{6} = 0.14675.$$

Note. If a negative characteristic is not exactly divisible by the index of the root, subtract from the characteristic the least positive number which will make it so divisible. Indicate the addition of the characteristic so formed to the mantissa, and prefix the number subtracted from the characteristic to the mantissa. Then divide separately. Thus,

$$\log \sqrt{.5} = \frac{\log .5}{2} = \frac{\bar{1}.6990}{2} = \frac{1.6990 + \bar{2}}{2} = 0.8495 + \bar{1} = \bar{1}.8495 \text{ or } 9.8495 - 10.$$

Exercise 110.

Given $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 5 = 0.6990$, $\log 7 = 0.8451$; find the logarithms of:

1. $\sqrt[5]{7}$; $\sqrt[3]{3}$; $\sqrt[5]{2}$; $\sqrt[6]{.5}$; $\sqrt[3]{243}$; $\sqrt[5]{12^3}$; $\sqrt[5]{.96}$; $\sqrt[n]{.07}$.

2. $\sqrt[3]{(16.2)^2}$; $5^{\frac{1}{3}} \times 3^{\frac{1}{4}}$; $\sqrt[5]{\frac{5}{28}}$; $\frac{\sqrt[5]{7}}{\sqrt[7]{5}}$; $\frac{\sqrt[3]{5} \times \sqrt[4]{3}}{\sqrt[7]{2}}$; $6^{\frac{1}{3}} \times 3^{\frac{3}{5}}$.

3. $\frac{\sqrt[4]{5} \times \sqrt[10]{2}}{\sqrt[3]{18} \times \sqrt{2}}$; $\frac{\sqrt[\frac{3}{2}]{42}}{\sqrt[\frac{2}{3}]{15}}$; $\frac{\sqrt[5]{5}}{\sqrt[7]{7}}$; $\sqrt[7]{\frac{(.21)^2}{(.00084)^2}}$; $\frac{\sqrt[10]{2\sqrt[\frac{1}{2}]{8}}}{\sqrt[6]{5\sqrt{15}}}$.

127. Table of Logarithms. The table (pages 304 and 305) gives the mantissæ of the logarithms to four decimal places for all numbers from 1 to 1000 inclusive. The **characteristic** and *decimal points are omitted,* and must be supplied by inspection (Art. 121.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Explanation of Table. The left-hand column, headed **N**, is a column of numbers. The figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, opposite **N** at the top of the table, are the right-hand figures of numbers whose left-hand figures are given in the column headed **N**. The figures in the column which they head are the corresponding mantissæ of the logarithms of the numbers.

128. To Find the Logarithm of a Number.

I. **Consisting of one Figure.** The mantissæ of the logarithms of single digits, 1, 2, 3, 4, etc., are seen opposite 10, 20, 30, 40, etc., and in the column headed 0. To the mantissa prefix the characteristic and insert the decimal point. Thus,

$$\log 6 = 0.7782. \quad \log .6 = \bar{1}.7782. \quad \log 8 = 0.9031.$$

Similarly, since the mantissa of log .009 is the same as the mantissa of log 9, $\log .009 = \bar{3}.9542$.

II. **Consisting of two Figures.** In the column headed **N** look for the figures. In the line with the figures, and in the column headed 0, is seen the mantissa. Then proceed as before. Thus,

$$\log 13 = 1.1139. \quad \log 2.5 = 0.3979. \quad \log .92 = \bar{1}.9638.$$

Similarly, $\log .00092 = \bar{4}.9638$.

III. **Consisting of three Figures.** In the column headed **N**, look for the first two figures, and at the top of the table for the third figure. In the line with the first two figures, and in the column headed by the third figure, is seen the mantissa. Then proceed as before. Thus,

$$\log 313 = 2.4955. \quad \log 17.9 = 1.2529. \quad \log .279 = \bar{1}.4456.$$

Similarly, $\log .000718 = \bar{4}.8561$.

IV. **Consisting of more than three Figures.** Take the mantissa of the logarithm of the first three figures as given in the table. Prefix a decimal point to the remaining figures of the number, and multiply the result by the tabular * difference. Add the product to

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* The *Tabular* difference is the difference between the two successive mantissæ between which the required, or given, mantissa lies.

the mantissa thus taken. Prefix the characteristic and insert the decimal point as before. Thus,

1. Find the logarithm of 80.672.

The tabular mantissa of the logarithm of 806 is	9063
The tabular mantissa of the logarithm of 807 is	9069
Therefore, the tabular difference =	6

The number 80672 being between 80600 and 80700, *the mantissa of its logarithm must be between* 9063 and 9069. An increase of 100 in 80600 causes an increase of 6 in the mantissa of the logarithm of 80600. Therefore, an increase of 72 in 80600 will produce an increase of $\frac{72}{100}$ of 6 (or $.72 \times 6$), or 4.32, in the mantissa of the logarithm of 80600. Hence, the tabular mantissa of log 80672 *must be* $9063 + 4$, or 9067. Prefixing the characteristic and inserting the decimal point, we have

$$\log 80.672 = 1.9067.$$

Similarly, since the mantissa of log .0005102 is the same as the mantissa of log 5102,

2. To find the logarithm of .0005102.

The tabular mantissa of log 510 is	7076
The tabular mantissa of log 511 is	7084
∴ the tabular difference =	8

Hence, the tabular mantissa of log 5102 *must be* $7076 + .2 \times 8$, or 7078.

$\therefore \log .0005102 = \bar{4}.7078.$

Exercise 111.

Find by means of the table the logarithms of the following:

1. 70; 102; 201; 999; .712; 3.6; .00789; 3.21.

2. .0031; .0983; .00003; 10.08; 29461; 3015.6.

3. 32678; $\sqrt{337}$; $\sqrt[5]{.0086672}$; $\frac{403}{959}$; $(.098 \times 85)^{\frac{5}{6}}$.

129. To Find a Number when its Logarithm is Given.

I. **If the Given Mantissa is Found in the Table.** The first two figures of the required number will be seen on the same line with the mantissa and in the column headed **N**, the third figure will be seen at the head of the column in which the mantissa is found. Finally insert the decimal point as the characteristic directs. Thus,

1. Find the number whose logarithm is $\bar{1}.9232$.

Look for 9232 in the table. It is found on the line with **83** and in the column headed **8**. Therefore, write 838 and insert the decimal point. Hence, the number required is .838.

II. **If the Given Mantissa cannot be Found in the Table.** Find the next less mantissa, and the corresponding number; also find the tabular difference. Annex the quotient of the difference between the given mantissa and the next less mantissa divided by the tabular difference, to the corresponding number; then proceed as before. Thus,

2. Find the number whose logarithm is $\bar{2}.7439$.

The next less mantissa is 7435, corresponding to 554.
The next greater mantissa is $\underline{7443}$, corresponding to 555.
$\therefore$ the tabular difference = 8.

The difference between the given mantissa and the next less mantissa is 4. Since the given mantissa lies between 7435 and 7443, the corresponding number *must lie between* 554 and 555. An increase of 8 in the mantissa causes an increase of 1 in 554. Therefore, an increase of 4 in the mantissa *will produce an increase* of $\frac{4}{8}$, or .5, in 554. Hence, the mantissa 7439 *must* correspond to the number $554 + .5$, or 554.5. Therefore (II, Art. 121), write 05545 and prefix the decimal point. Hence, the number required is .05545.

3. Find the number whose logarithm is 3.1658.

The next less mantissa is 1644, corresponding to 146.
The next greater mantissa is $\underline{1673}$, corresponding to 147.
$\therefore$ the tabular difference = 29.

The difference between the given mantissa and the next less mantissa is 14. Annex $\frac{14}{29}$, or .48 nearly, to the number 146, and insert the decimal point as the characteristic directs. Hence, the number required is 1464.8.

Exercise 112.

Find the numbers whose logarithms are:

1. 3.4683; $\bar{2}.4609$; 4.8055; 0.4984; 0.1959.

2. $\bar{3}.6580$; 2.4906; 4.5203; $\bar{2}.5228$; 0.6595.

3. 0.8800; 1.7038; 5.8017; $\bar{3}.1144$; 5.7319.

130. An **Exponential Equation** is one in which the exponent is the unknown number; as, $m^x = n$, $m^{\frac{1}{x}} = n$. Such equations usually require logarithms for their solutions.

Example 1. Solve the equation $21^x = 1.5$.

Process. Take the logarithm of each member, $x \log 21 = \log 1.5$.
By means of the table, $1.3222\,x = .1761$.

Therefore, $x = \dfrac{.1761}{1.3222} = .1332$, nearly.

Example 2. Find the value of $3.208 \times .0362 \times .15734$.

Process. $\log(3.208 \times .0362 \times .15734) = \log 3.208 + \log .0362 + \log .15734$.

$$\begin{aligned}
\log 3.208 &= 0.5062\\
\log .0362 &= \bar{2}.5587\\
\log .15734 &= \underline{\bar{1}.1969}\\
&\ \bar{2}.2618 = \log .01827.
\end{aligned}$$

Therefore, $3.208 \times .0362 \times .15734 = .01827$.

Example 3. Find the fifth root of $.05678$.

Process. $\log .05678 = \bar{2}.7542$.

$5\,)\,\bar{2}.7542 = 5\,)\,\underline{3.7542 + \bar{5}}$

$\qquad\qquad .7508 + \bar{1} = \bar{1}.7508 = \log .5634$, nearly.

Therefore, $\sqrt[5]{.05678} = .5634$, nearly.

EXAMPLE 4. Find the value of $\log_{2\sqrt{3}} 144$.

Solution. To find $\log_{2\sqrt{3}} 144$, is the same as solving (Art. 119) $(2\sqrt{3})^l = 144$, for l, squaring each side, etc., $l = 4$.

Therefore, $\log_{2\sqrt{3}} 144 = 4$.

Exercise 113.

Find by logarithms the values of the following:

1. $360 \times .0827$; $117.57 \times .0404$; $\dfrac{104.8}{37.25}$; $(31.89)^3$.

2. $\sqrt[7]{951}$; $380.57 \times .000967$; $\dfrac{\sqrt[3]{(.0265)^2} \times \sqrt[5]{.0009163}}{.28538}$.

3. $\dfrac{212.6 \times 30.2}{84.3 \times 3.62 \times .05632}$; $\dfrac{7435}{38731 \times .3962}$; $\dfrac{\sqrt[5]{343}}{\sqrt[4]{729}}$.

4. $\dfrac{\sqrt[3]{9.675} \times \sqrt[5]{21.6}}{\sqrt[7]{385.67}}$; $72132 \times .038209$; $\sqrt[11]{.000313}$.

5. $(61173)^{\frac{5}{6}}$; $\dfrac{\sqrt[9]{112}}{(.19268)^{\frac{2}{5}}}$; $\dfrac{5\sqrt[3]{2}}{\sqrt[6]{27}}$; $\sqrt[8]{3} \times \sqrt[7]{.001}$; $\dfrac{3\sqrt[5]{4}}{5\sqrt[3]{49}}$.

Solve the following equations:

6. $20^x = 100$; $2^x = 769$; $10^x = 4.4$; $(\frac{5}{3})^x = 17.4$.

7. $10^{\frac{1}{x}} = 2.45$; $5^{5-3x} = 2^{x+2}$; $\sqrt[x]{3^{5x-8}} = \sqrt[7x]{9^{3x+1}}$.

8. $2^x \times 6^{x-2} = 5^{2x} \times 7^{1-x}$; $3^{x-2} = 5$; $4^x = 64$.

9. $(\frac{1}{2})^x = 10$; $m^x = n$; $m^{ax+b} = n$; $m^{ax} \times c^{bx} = n$.

10. $2^{x+y} = 6^y$, $3^x = 3 \times 2^{y+1}$; $3^{1-x-y} = 4^{-y}$, $2^{2x-1} = 3^{3y-x}$.

11. $a^{2x}n^{3y} = m^5$, $a^{3x}n^{2y} = m^{10}$; $m^x m^{5y} = (m^7)^4$, $\dfrac{m^{7x}}{m^6} = (m^y)^3$.

Find the number of digits in the values of:

12. $3^{12} \times 2^8$; 2^{64}; 16^{100}; $(4375)^8$; $(396000)^{10}$.

Find the number of ciphers between the decimal point and the first significant figure in the values of:

13. $(.2)^4$; $(.5)^{100}$; $(.05)^5$; $(.0336)^{10}$; $\sqrt[-3]{8100}$.

14. Given $\log x = \bar{2}.30103$, find $\log x^{\frac{1}{7}}$.

Find the values of:

15. $\log_2 4$; $\log_2 8$; $\log_2 32$; $\log_2 128$; $\log_2 1024$.

16. $\log_2 \frac{1}{4}$; $\log_2 \frac{1}{8}$; $\log_2 \frac{1}{32}$; $\log_2 \frac{1}{64}$; $\log_2 \sqrt[3]{16}$.

17. $\log_3 729$; $\log_5 125$; $\log_5 625$; $\log_5 15625$; $\log_4 \frac{1}{4}$.

18. $\log_{-6} 1296$; $\log_{-6} -\frac{1}{216}$; $\log_6 \frac{1}{216}$; $\log_{2\sqrt{2}} 512$.

19. $\log_{5\sqrt{5}} 125$; $\log_{343} 49$; $\log_8 128$; $\log_{2\sqrt{3}} \frac{1}{144}$.

20. $\log_{2\sqrt{2}} \frac{1}{256}$; $\log_{27} \frac{1}{81}$; $\log_{\frac{1}{2}} 4$; $\log_a a$; $\log_a \dfrac{1}{a}$.

21. If 8 is the base, of what number is $\frac{2}{3}$ the logarithm? Of what $\frac{4}{3}$? Of what 1? Of what 2? Of what 3? Of what $1\frac{2}{3}$? Of what $2\frac{1}{3}$? Of what $3\frac{1}{3}$? Of what $\frac{11}{3}n$?

22. In the systems whose bases are 10, 3, and $\frac{1}{3}$, of what numbers is -5 the logarithm?

Find the bases of the systems in which:

23. $\log 81 = 4$; $\log 81 = -4$; $\log \frac{81}{10000} = 4$; $\log \frac{81}{10000} = -4$; $\log \frac{100}{81} = \pm 2$; $\log 1024^n = \pm 5n$.

CHAPTER XXI.

QUADRATIC EQUATIONS.

131. A **Quadratic Equation** is an equation in which the square is the highest power of the unknown number.

A **Pure Quadratic Equation** is an equation which contains only the square of the unknown number; as, $5x^2 = 17$.

An **Affected Quadratic Equation** is an equation which contains both the square and the first power of the unknown number; as, $5x^2 - 2x = 10$.

EXAMPLE. Solve $\dfrac{x^2+5}{x} - \frac{3}{4}x = \dfrac{x}{3} + \dfrac{17}{4x}$.

Process. Clearing of fractions, $12x^2 + 60 - 9x^2 = 4x^2 + 51$.

Transposing and uniting, $x^2 = 9$.

Therefore, extracting the square root,* $x = \pm 3$. Hence,

To Solve a Pure Quadratic Equation. Find the value of the square of the unknown number by the method for solving a simple equation, and then extract the square root of both members.

Note. * In extracting the square root of both members of the equation $x^2 = 9$, we ought to prefix the double sign ($\pm$) to the square root of each member; but there are no new results by it, and it is sufficient to write the double sign before one member only. Thus, if we write $\pm x = \pm 3$, we have $+x = +3$, $+x = -3$, $-x = +3$, and $-x = -3$; but the last two become identical to the first two on changing the signs of both members. So that in either case, $x = 3$, and $x = -3$.

Exercise 114.

Solve the following equations:

1. $x^2 + 5 = \frac{10}{3}x^2 - 16$; $\frac{3}{4}x^2 - (2x^2 - 3) = \dfrac{16x^2 + 9}{5}$.

2. $8x + \dfrac{7}{x} = \frac{65}{7}x$; $15x - \dfrac{11}{x} = \frac{13}{5}x$.

3. $x(x - 10) = (6\frac{2}{5} - x)10$; $(5x + \frac{1}{2})^2 = 756\frac{1}{2} + 5x$.

4. $\dfrac{35 - 2x}{9} + \dfrac{5x^2 + 7}{5x^2 - 7} = \dfrac{17 - \frac{2}{3}x}{3}$.

5. $\dfrac{8}{1 - 2x} + \dfrac{8}{1 + 2x} = 25$; $9x^2 = 16(2 - x^2)$.

6. $\dfrac{7x^2 + 8}{21} - \dfrac{x^2 + 4}{8x^2 - 11} = \dfrac{x^2}{3}$; $(5x - \frac{3}{2})^2 = \frac{5}{2} - 15x$.

7. $\frac{6}{5}(2x - 5)^2 = 94 - 24x$; $3x^2 - 4 = \dfrac{x^2 + 2}{5x^0}$.

8. $\dfrac{m}{x^2 - n} = \dfrac{n}{x^2 - m}$; $ax^2 + mx = ac^2 + mx$.

9. $x^2 + mx - n = mx(1 - mx)$; $2 + 4x^2 = a(1 - x^2)$.

10. $x^2 - nx + m = nx(x - 1)$; $x\sqrt{6 + x^2} = 1 + x^2$.

11. $\dfrac{mn - x}{n - mx} = \dfrac{n - ax}{an - x}$; $x + \sqrt{x^2 - 3} = \dfrac{5}{\sqrt{x^2 - 3}}$.

12. $\dfrac{a}{b + x} + \dfrac{a}{b - x} = c$; $\dfrac{1}{1 - \sqrt{1 - x^2}} - \dfrac{1}{1 + \sqrt{1 - x^2}} = \dfrac{\sqrt{3}}{x^2}$.

132. Example 1. Solve $x^2 - 2ax + 4ab = 2bx$.

Solution. Transposing, we have $x^2 - 2ax + 4ab - 2bx = 0$. Arrange in binomial terms and factor, and we have $(x - 2a)(x - 2b) = 0$. A product cannot be zero unless one of the factors is zero. Hence, the equation is satisfied if $x - 2a = 0$, or $x - 2b = 0$; that is, if $x = 2a$, or if $x = 2b$.

Example 2. Solve $\frac{1}{2}x^2 + \frac{2}{3}x + 20\frac{1}{2} = 42\frac{2}{3} + x$.

Process. Clear of fractions, transpose, and unite,

$$3x^2 - 2x - 133 = 0.$$

Factor, $$(3x + 19)(x - 7) = 0.$$

Therefore, $3x + 19 = 0$, and $x - 7 = 0$.
Hence, $x = -6\frac{1}{3}$, and $x = 7$.

To Solve a Quadratic Equation by Factoring. Simplify the equation, with all its terms in the first member; then place the factors of the first member separately equal to zero, and solve the simple equations thus formed.

Exercise 115.

Solve the following equations:

1. $x^2 - 10x = 24$; $x^2 + 2x = 80$; $x^2 - 18x + 32 = 0$.

2. $x^2 + 10 = 13(x + 6)$; $x^2 + 4x - 50 = 2 - 5x$.

3. $4x^2 + 13x + 3 = 0$; $3x^2 + 1 = -11x - x^2 + 4$.

4. $x^2 - x = 11342$; $5x^2 + 3x - 4 = 8x - 7x^2 - 2$.

5. $1 - 3x - x^2 = 2x^2 + x - 3$; $x^2 - 2ax + 8x = 16a$.

6. $x^3 - 5x^2 = 5x^3 + 7x^2$; $x^2 - \frac{9}{5}x + \frac{9}{20} = 0$.

7. $11x^2 - 11\frac{1}{4} = 9x$; $\dfrac{x + 3}{x + 2} - \dfrac{2x - 3}{x - 1} = \dfrac{3 - x}{x - 2}$.

8. $(x-2)(x^2+9x+20)=0;\;\; 2x^3+3x^2-2x-3=0.$

9. $\dfrac{21x^3-16}{3x^2-4}-3x+1=6+4x;\;\; \dfrac{2x(a-x)}{3a-2x}=\dfrac{a}{4}.$

10. $mqx^2-mnx+pqx-np=0;\;\; x+5=\sqrt{x+5}+6.$

11. $(a-b)x^2-(a+b)x+2b=0;\;\; x^2=21+\sqrt{x^2-9}.$

133. An affected quadratic equation can always be solved by the method of *completing the square*. This method consists in adding to both members such an expression as will make the member, with all the terms containing the unknown number, a perfect square. The explanation of this method depends upon the principle that a trinomial is a perfect square when one of its terms is *plus* or *minus twice* the product of the square roots of the other two. This process enables us to extract the square root of the member containing the unknown number, and thus form two simple equations which may be solved separately.

EXAMPLE. Solve $\frac{1}{2}(8-x)-\dfrac{2x-11}{x-3}=\frac{1}{6}(x-2).$

Process. Clear of fractions, transpose, and unite,

$$-4x^2+26x=12.$$

Divide by -4, $$x^2-\tfrac{13}{2}x=-3.$$

Add * $(\frac{13}{4})^2$ to both members, $x^2-\frac{13}{2}x+(\frac{13}{4})^2=-3+(\frac{13}{4})^2=\frac{121}{16}.$

Extract the square root, $$x-\tfrac{13}{4}=\pm\tfrac{11}{4}.$$

Therefore, $x-\frac{13}{4}=\frac{11}{4}$, and $x-\frac{13}{4}=-\frac{11}{4}$. $x=6$, and $x=\frac{1}{2}$.

Every affected quadratic equation may be reduced to the *general* form

$$mx^2+nx+a=0;$$

where m, n, and a represent any numbers whatever, positive or negative, integral or fractional. Dividing both members by m for convenience, representing $\dfrac{a}{m}$ by b and $\dfrac{n}{m}$ by c, and transposing, we have

$$x^2 + cx = -b.$$

Add $^*\left(\frac{c}{2}\right)^2$ to both members, $x^2 + cx + \left(\frac{c}{2}\right)^2 = -b + \left(\frac{c}{2}\right)^2.$

Or, $x^2 + cx + \left(\frac{c}{2}\right)^2 = \frac{1}{4}(c^2 - 4b).$

Extract the square root, $x + \frac{c}{2} = \pm \frac{1}{2}\sqrt{c^2 - 4b}.$

Therefore, $x + \frac{c}{2} = \frac{1}{2}\sqrt{c^2 - 4b}$, and $x + \frac{c}{2} = -\frac{1}{2}\sqrt{c^2 - 4b}$. From which $x = -\frac{c}{2} + \frac{1}{2}\sqrt{c^2 - 4b}$, and $x = -\frac{c}{2} - \frac{1}{2}\sqrt{c^2 - 4b}$. These values may be written in the form $x = \frac{-c \pm \sqrt{c^2 - 4b}}{2}$. Hence,

Common Method of Solving Quadratics. *Reduce the equation to the form* $x^2 + cx = -b$. *Complete the square of the first member by adding to each member of the equation the square of half the coefficient of* x. *Extract the square root of both members, and solve the resulting simple equations.*

Notes: 1. * Always indicate the square of the expression to be added, in the first member.

2. Since the squared terms of the square of a binomial are *always positive*, the coefficient of x^2 must be made $+1$, if necessary, before completing the square. This may be done by multiplying or dividing both members by -1.

3. The foregoing method is called the **Italian Method**, having been used by Italian mathematicians, who first introduced a knowledge of algebra into Europe.

134. It is often convenient to complete the square without first reducing the simplified equation to the form in which the coefficient of x^2 is 1. Thus,

Example 1. Solve $\frac{3x - 7}{x} + \frac{4x - 10}{x + 5} = \frac{7}{2}.$

Process. Clear of fractions,

$$6x^2 - 14x + 30x - 70 + 8x^2 - 20x = 7x^2 + 35x.$$

Transpose and unite, $\qquad 7x^2 - 39x = 70.$

Multiply by 7×4, $\qquad 196x^2 - 1092x = 1960.$

Add $(39)^2$, $\qquad 196x^2 - 1092x + (39)^2 = 1960 + 1521 = 3481.$

Extract the square root, $\qquad 14x - 39 = \pm 59.$

Therefore, $14x = 39 + 59$, and $14x = 39 - 59$. $x = 7$, and $x = -\frac{10}{7}$.

Verify by putting these numbers for x in the original equation.

Process. $x = 7$.

$$\frac{3 \times 7 - 7}{7} + \frac{4 \times 7 - 10}{7 + 5} = \frac{7}{2},$$

$$2 + \frac{3}{2} = \frac{7}{2},$$

$$\frac{7}{2} = \frac{7}{2}.$$

$x = -\frac{10}{7}$.

$$\frac{3 \times -\frac{10}{7} - 7}{-\frac{10}{7}} + \frac{4 \times -\frac{10}{7} - 10}{-\frac{10}{7} + 5} = \frac{7}{2},$$

$$\frac{-\frac{79}{7}}{-\frac{10}{7}} + \frac{-\frac{110}{7}}{\frac{25}{7}} = \frac{7}{2},$$

$$\frac{79}{10} - \frac{22}{5} = \frac{7}{2},$$

$$\frac{7}{2} = \frac{7}{2}.$$

When a quadratic equation appears in the general form

$$mx^2 + nx + a = 0,$$

the terms containing x may be made a complete square, without first dividing the equation by the coefficient of x^2. Thus,

Transpose a, $\qquad mx^2 + nx = -a$

Multiply the equation by $4m$ and add the square of n,

$$4m^2x^2 + 4mnx + n^2 = n^2 - 4am.$$

Extract the square root, $\qquad 2mx + n = \pm\sqrt{n^2 - 4am}.$

Transpose n, $\qquad 2mx = -n \pm \sqrt{n^2 - 4am}$

Therefore, $x = \dfrac{-n \pm \sqrt{n^2 - 4am}}{2m}$. Hence,

Hindoo Method of Solving Quadratics. *Reduce the equation to the form* $mx^2 + nx = -a$. *Multiply it by four times the coefficient of* x^2, *and complete the square by adding to each member the square of the coefficient of* x *in the given equation. Extract the square root of both members, and solve the resulting simple equations.*

If the coefficient of x in the given equation is an *even* number, the square may be completed as follows:

Multiply the equation by the coefficient of x^2, *and add to each member the square of half the coefficient of* x *in the given equation.*

EXAMPLE 2. Solve $\frac{8x}{x+2} - \frac{20}{3x} = 6$.

Process. Free from fractions,

$$(3x)(8x) - 20(x+2) = 6(3x)(x+2).$$

Simplify, $\quad 6x^2 - 56x = 40.$

Multiply by 6, $\quad 36x^2 - 336x = 240.$

Add $(\frac{56}{2})^2$, $\quad 36x^2 - 336x + (28)^2 = 1024.$

Extract the square root,* $\quad 6x - 28 = \pm 32.$

Transpose, $\quad 6x = 28 + 32$, or $28 - 32$.

Therefore, $\quad x = 10$, or $-\frac{2}{3}$.

Verify by substituting 10 for x in the original equation.

Process.

$$\frac{8 \times 10}{10+2} - \frac{20}{3 \times 10} = 6,$$

$$\frac{20}{3} - \frac{2}{3} = 6,$$

$$6 = 6.$$

Verify by substituting $-\frac{2}{3}$ for x in the original equation.

Process.

$$\frac{8 \times -\frac{2}{3}}{-\frac{2}{3} + 2} - \frac{20}{3 \times -\frac{2}{3}} = 6,$$

$$\frac{-\frac{16}{3}}{\frac{4}{3}} - \frac{20}{-2} = 6,$$

$$-4 + 10 = 6,$$

$$6 = 6.$$

Notes: 1. * We ought to write the double sign before the root of both members. Thus, $6x - 28 = \pm 32$, the reason for not doing so is the same as given in Art. 131, Note.

2. The **Hindoo**, or **Indian Method**, is supposed to have been discovered by Aryabhalta, a celebrated Hindoo mathematician, and one of the first inventors of algebra. It is not only more general in form, but much better adapted to the solution of equations in which the coefficient of the square of the unknown number is not 1.

3. This method has an advantage over the common method in avoiding fractions in completing the square, and is often preferred in solving literal equations.

135. In case the coefficient of the square of the unknown, in the simplified equation, is a ***square number*** the square may be completed as follows:

EXAMPLE 1. Solve $72x - 54 = (20 - x)(4x + 3)$.

Process. Simplify, $4x^2 - 5x = 114$.

Add $\left(\dfrac{5x}{2\sqrt{4x^2}}\right)^2$, or $\left(\dfrac{5}{4}\right)^2$, $4x^2 - 5x + (\frac{5}{4})^2 = \frac{1849}{16}$.

Extract the root, $2x - \frac{5}{4} = \pm\frac{43}{4}$.

Transpose, $2x = \frac{5}{4} + \frac{43}{4}$, or $\frac{5}{4} - \frac{43}{4}$.

Therefore, $x = 6$, or $-4\frac{3}{4}$.

The coefficient of x^2 may always be made a *square number* by multiplication or division. Hence,

General Method of Solving Quadratics. *Add to each member the square of the quotient obtained from dividing the second term by twice the square root of the first term. Then proceed as before.*

EXAMPLE 2. Solve $\dfrac{5}{x+4} + \dfrac{3}{x} = \dfrac{35}{x-2}$.

Process. Free from fractions,

$$5(x-2)x + 3(x+4)(x-2) = 35(x+4)x.$$

Simplify, $-27x^2 - 144x = 24$.

Divide by -3, $9x^2 + 48x = -8$.

Add $\left(\dfrac{48x}{2\sqrt{9x^2}}\right)^2$, or $(8)^2$, $9x^2 + 48x + (8)^2 = 56$.

Extract the root, $3x + 8 = \pm 2\sqrt{14}$.

Transpose, $3x = -8 \pm 2\sqrt{14}$. $\therefore\ x = \dfrac{-8 \pm 2\sqrt{14}}{3}$.

Note. The Common and Hindoo Methods of completing the square are modifications of the General Method.

Exercise 116.

Solve the following:

1. $23x = 120 + x^2$; $42 + x^2 = 13x$; $12x^2 + x = 1$.

2. $22x + 23 - x^2 = 0$; $x^2 - \frac{2}{3}x = 32$; $x^2 + 3x = 4$.

3. $x + 22 - 6x^2 = 0$; $25x = 6x^2 + 21$; $x^2 - 2x = 3$.

4. $3x^2 + 121 = 44x$; $\frac{19}{5}x = \frac{4}{5} - x^2$; $91x^2 - 2x = 4$.

5. $21x^2 + 22x + 5 = 0$; $9x^2 - 143 - 6x = 0$.

6. $18x^2 - 27x - 26 = 0$; $50x^2 - 15x = 27$.

7. $19x = 15 - 8x^2$; $x^2 + \frac{4}{15}x = \frac{1}{5}$; $x^2 - \frac{7}{6}x - \frac{1}{2} = 0$.

8. $5x^2 + 14x = 55$; $(x + 1)(2x + 3) = 4x^2 - 22$.

9. $2(x - 3) = 3(x + 2)(x - 3)$; $.3x^2 + 2.1x + 1 = 0$.

10. $25x + 2x^2 = .0025$; $\frac{3}{5}(x + 6)(x - 2) = \frac{2}{3}(62\frac{1}{10} + \frac{18}{5}x)$.

11. $\dfrac{1}{1+x} - \dfrac{1}{3-x} = \dfrac{6}{35}$; $\dfrac{x+16}{5} + \dfrac{11}{x} = \dfrac{4x - 17\frac{1}{3}}{3}$.

12. $\dfrac{4x}{9} + \dfrac{x-5}{x+3} = \dfrac{4x+7}{19}$; $\dfrac{x+4}{x-4} + \dfrac{x-2}{x-3} = 6\frac{1}{3}$.

13. $\dfrac{1}{3-x} - \dfrac{4}{5} = \dfrac{1}{9-2x}$; $\dfrac{4}{x-3} - \dfrac{3}{x+5} = \dfrac{1}{18}$.

14. $\dfrac{5}{x-2}-\dfrac{4}{x}=\dfrac{3}{x+6}$; $\dfrac{4}{x-1}-\dfrac{5}{x+2}=\dfrac{3}{x}$.

15. $x+\dfrac{x^2+3}{x^2-5}=\dfrac{12+5x^3}{5(x^2-5)}$; $\dfrac{3x}{x+1}+\dfrac{2x-5}{3x-1}=3\tfrac{10}{69}$.

16. $\dfrac{5x-7}{7x-5}=\dfrac{x-5}{2x-13}$; $\dfrac{3x-1}{4x+7}=1-\dfrac{6}{x+7}$.

17. $\dfrac{12x^3-11x^2+10x-78}{8x^2-7x+6}=1\tfrac{1}{2}x-\tfrac{1}{2}$.

18. $\dfrac{3x+5}{3x-5}-\dfrac{3x-5}{3x+5}=\dfrac{135}{176}$; $\dfrac{7}{x^2+4x}+\dfrac{21}{3x^2-8x}=\dfrac{22}{x}$.

19. $\dfrac{6}{x-1}-\dfrac{18}{x+5}=\dfrac{7}{x+1}-\dfrac{8}{x-5}$; $\dfrac{x}{x+8}=\dfrac{x+3}{2x+1}$.

136. Literal Quadratic Equations.

EXAMPLE 1. Solve $mx^2+nx=\dfrac{mn}{m+n}+mx-nx^2$.

Process. Transpose and factor, $(m+n)x^2-(m-n)x=\dfrac{mn}{m+n}$.

Multiply the equation by $4(m+n)$ and add the square of $(m-n)$,

$$4(m+n)^2x^2-4(m^2-n^2)x+(m-n)^2=(m+n)^2.$$

Extract the square root, $2(m+n)x-(m-n)=\pm(m+n)$.

Transpose, $2(m+n)x=m-n\pm(m+n)$

$= 2m$, or $-2n$.

Therefore, $x=\dfrac{m}{m+n}$, or $-\dfrac{n}{m+n}$.

EXAMPLE 2. Solve $\dfrac{2x+1}{b}-\dfrac{1}{x}\left(\dfrac{1}{b}-\dfrac{2}{a}\right)=\dfrac{3x+1}{a}$.

Process. Free from fractions,

$$a\,x\,(2\,x+1)-(a-2\,b)=b\,x\,(3\,x+1).$$

Simplify and transpose,

$$2\,a\,x^2+a\,x-3\,b\,x^2-b\,x=a-2\,b.$$

Express the first member in *two terms*,

$$(2\,a-3\,b)\,x^2+(a-b)\,x=a-2\,b.$$

Multiply by $4\,(2\,a-3\,b)$,

$$4\,(2\,a-3\,b)^2x^2+4\,(2\,a-3\,b)\,(a-b)\,x=8\,a^2-28\,a\,b+24\,b^2.$$

Complete the square,

$$4(2\,a-3\,b)^2x^2+4(2\,a-3\,b)(a-b)x+(a-b)^2=9\,a^2-30\,a\,b+25\,b^2.$$

Extract the square root,

$$2\,(2\,a-3\,b)\,x+(a-b)=\pm\,(3\,a-5\,b).$$

Transpose, $$2\,(2\,a-3\,b)\,x=-\,(a-b)\pm(3\,a-5\,b)$$
$$=2\,a-4\,b,\text{ or }-2(2\,a-3\,b.$$

Therefore, $$x=\frac{a-2\,b}{2\,a-3\,b},\text{ or }-1.$$

EXAMPLE 3. Solve $\dfrac{1}{x}+\dfrac{1}{b+x}=\dfrac{1}{a}+\dfrac{1}{a+b}$.

Process. Transpose, $$\frac{1}{x}-\frac{1}{a}=\frac{1}{a+b}-\frac{1}{b+x}.$$

Reduce each member to a common denominator,

$$\frac{a-x}{a\,x}=\frac{x-a}{(a+b)\,(b+x)}.$$

Free from fractions, $(a-x)\,(a+b)(b+x)=a\,x\,(x-a)$.

Transpose and factor,

$$(a-x)\,[(a+b)\,(b+x)+a\,x]=0.$$

Hence (Art. 11), $a-x=0.\quad\therefore\ x=a.$

Also, $(a+b)\,(b+x)+a\,x=0.$

Simplify and factor,

$$b\,(a+b)+(2\,a+b)\,x=0.\quad\therefore\ x=-\frac{b\,(a+b)}{2\,a+b}.$$

Note. Always express the first member of the simplified equation in two terms, the first term involving x^2, the second involving x.

Exercise 117.

Solve the following equations :

1. $x^2-(a+b)x+ab=0$; $mx^2+\frac{mbx}{a}-m=\frac{max}{b}$.

2. $\frac{x}{a}+\frac{a}{x}=\frac{x}{b}+\frac{b}{x}$; $mx^2-\frac{mbx}{a}-m=-\frac{amx}{b}$.

3. $(a-b)x^2-(a+b)x+2b=0$; $a^2x^2+abx=2b^2$.

4. $\frac{x^2}{a^2}+\frac{x}{b}=\frac{2a^2}{b^2}$; $\frac{2x(a-x)}{3a-2x}=\frac{a}{4}$; $\frac{x^2+m^2}{x}=5$.

5. $mqx^2-mnx+pqx-np=0$; $x^2+2x\sqrt{n}=n$.

6. $a^2x^2-2a^3x+a^4-1=0$; $x^2+x(a-b)=ab$.

7. $x^2+mx+cx+n^{\frac{1}{2}}x+m^2x=0$.

8. $4a^2x=(a^2-b^2+x)^2$; $a^2(x-a)^2=b^2(x+a)^2$.

9. $\left(\frac{x}{a}-\frac{2a}{x}-1\right)\left(1+\frac{a}{x}-\frac{2x}{a}\right)=0$.

10. $\frac{x^2-4abx}{(a+b)^2}=(a-b)^2$; $x^2-\frac{a}{b}x-\frac{m}{n}x=-cx-x$.

11. $\frac{(a^2-b^2)(x^2+1)}{a^2+b^2}=2x$; $9a^4b^4x^2-6a^3b^2x=b^2$.

12. $\frac{nx+a}{x-b}=\frac{x+b}{nx-a}$; $\frac{1}{a}+\frac{1}{a+x}+\frac{1}{a+2x}=0$.

13. $\dfrac{a-x}{x}+\dfrac{x}{a-x}=\dfrac{b}{c}$; $\dfrac{(a-1)^2x^2+2(3a-1)x}{4a-1}=1.$

14. $\left(x+\dfrac{1}{x}\right)^2=4x^2$; $\left(ax-\dfrac{a}{x}\right)^2=\frac{1}{4}a^2x^2.$

137. Solution by a Formula. From the quadratic equation $mx^2+nx=-a$,

$$x=\frac{-n\pm\sqrt{n^2-4am}}{2m} \tag{1}$$

By means of this formula the values of x, in an equation of the general form, may be written at once. Thus,

EXAMPLE 1. Solve $10x^2-23x=-12$.

Process. Here, $m=10$, $n=-23$, and $a=12$.

Substitute these values in (1), $x=\dfrac{-(-23)\pm\sqrt{(-23)^2-4\times12\times10}}{2\times10}$

$=\dfrac{23\pm7}{20}$

$=\frac{3}{2}$, or $\frac{4}{5}$.

EXAMPLE 2. Solve $\dfrac{1}{b+c+y}=\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{y}$.

Process. Free from fractions, transpose, and factor,

$$(b+c)y^2+(b+c)^2y=-bc(b+c).$$

Divide by $b+c$, $y^2+(b+c)y=-bc$.

Here, $m=1$, $n=b+c$, and $a=bc$.

Substitute these values in (1), $x=\dfrac{-(b+c)\pm\sqrt{(b+c)^2-4bc}}{2}$

$=\dfrac{-(b+c)\pm(b-c)}{2}$

$=-c$, or $-b$.

Note. In substituting the student must pay particular attention to the *signs* of the coefficients.

Miscellaneous Exercise 118.

1. $17x^2 + 19x = 1848$; $3x^2 - 12x + 1 = 6x - 23$.

2. $5x^2 + 4x = 273$; $\frac{5}{7}x^2 + \frac{7}{5}x + \frac{73}{140} = 0$.

3. $\dfrac{3}{x^2 - 3x} + \dfrac{6}{2x^2 + 8x} = \dfrac{27}{8x}$; $\frac{3}{5}(x+3)^2 = \frac{2}{3}(x+3)^2 - \frac{5}{3}$.

4. $x + \dfrac{4}{5x} + \dfrac{x+1}{5} = 3$; $\dfrac{x+2}{x-1} = \dfrac{2x+16}{x+5} - \dfrac{x-2}{x+1}$.

5. $16x^2 - 6x - 1 = 0$; $\dfrac{2x+3}{2(2x-1)} - \dfrac{7-x}{2(x+1)} = \dfrac{7-3x}{4-3x}$.

6. $\dfrac{a}{3} + \dfrac{5x}{4} = \dfrac{x^2}{3a}$; $\dfrac{x-8}{x-5} + \dfrac{2(x+8)}{x+4} = \dfrac{3x+10}{x+1}$.

7. $\dfrac{m + nx^2}{x} = \dfrac{x}{a}$; $\frac{16}{25}(5x^2 + 36)^2 = \frac{9}{16}(8x^2 - 4)^2$.

8. $\dfrac{1}{p+x} = \dfrac{1}{p} + \dfrac{1}{x}$; $11x^2 + 10ax = \pm a^2$; $x + \dfrac{1}{x} + 2 = \dfrac{n}{x}$.

9. $ax + dx^2 - a = d$; $\dfrac{x^2}{m^2} + \dfrac{bx}{c} = \dfrac{a^2}{m^2} + \dfrac{ab}{c}$.

10. $mx^2 - \dfrac{(m^3 - n^2)x}{mn} = 1$; $\dfrac{x^2}{3m - 2a} - \dfrac{x}{2} = \dfrac{m^2 - 4a^2}{4a - 6m}$.

11. $x^2 - 2ax = (b - c + a)(b - c - a)$.

12. $x^2 - (a + b)x = \frac{1}{4}(m + n + a + b)(m + n - a - b)$.

13. $\dfrac{1}{a+x} \pm \dfrac{1}{a-x} = \dfrac{a^2 + x^2}{a^2 - x^2}$; $\dfrac{x+3}{x-3} - \dfrac{x-3}{x+3} = a$.

14. $a b x^2 - 2 x (a + b) \sqrt{a b} = (a - b)^2$.

15. $x^2 + \dfrac{a - b}{a b^2} = \dfrac{14 a^2 - 5 a b - 10 b^2}{18 a^2 b^2} + \dfrac{(2 a - 3 b) x}{2 a b}$.

16. $\dfrac{a - 2 b - x}{a^2 - 4 b^2} - \dfrac{5 b - x}{a x + 2 b x} + \dfrac{2 a - x - 19 b}{2 b x - a x} = 0$.

17. $\dfrac{1}{2x^2 + x - 1} + \dfrac{1}{2 x^2 - 3x + 1} = \dfrac{m}{2nx - n} - \dfrac{2 n x + n}{m x^2 - m}$.

18. $\dfrac{1}{a x - \sqrt{a^2 x^2 - 1}} - \dfrac{1}{a x + \sqrt{a^2 x^2 - 1}} = a x$.

Query. What is the difference between the meaning of "the root of an equation" and "the root of a number"?

138. Problems. The following problems lead to pure or affected quadratic equations of one unknown number. In solving such problems, the equations of conditions will have two solutions. Sometimes both will fulfill the conditions of the problem; but generally one only will be a solution.

Exercise 119.

1. Find a number whose square diminished by 119 is equal to 10 times the excess of the number over 8.

Solution. Let $x =$ the number.
Then, $x - 8 =$ the excess of the number over 8.
Therefore, $x^2 - 119 = 10 (x - 8)$.
The solution of which gives, $x = 13$, or -3.
Only the positive value of x is admissible. Hence, the number is 13.

Note. In solving problems involving quadratics, the student should retain only those values for results that will satisfy the conditions of the problem.

2. The difference of the squares of two consecutive numbers is 17. Find the numbers.

3. Find two numbers whose sum is 9 times their difference, and the difference of whose squares is 81.

4. Find two numbers, such that their product is 126, and the quotient of the greater divided by the less is $3\frac{1}{2}$.

5. Divide 14 into two parts, such that the sum of the quotients of the greater divided by the less and of the less by the greater may be $2\frac{1}{12}$.

6. Find two numbers whose product is m, and the quotient of the greater divided by the less is n.

7. Find a number which when increased by n is equal to m times the reciprocal of the number. Find the number when $n = 17$ and $m = 60$.

8. Divide m into two parts, so that the sum of the two fractions formed by dividing each part by the other may be n. Solve when $m = 35$ and $n = 2\frac{1}{12}$.

9. Divide a into two parts, so that n times the greater divided by the less shall equal m times the less divided by the greater. Solve when $a = 14$, $n = 9$, and $m = 16$.

10. A farmer bought some sheep for \$72, and found that if he had received 6 more for the same money, he would have paid \$1 less for each. How many did he buy?

11. If a train travelled 5 miles an hour faster, it would take one hour less to travel 210 miles. Find the rate travelled and number of hours required.

12. A man travels 108 miles, and finds that he could have made the journey in $4\frac{1}{2}$ hours less had he travelled 2 miles an hour faster. Find the rate he travelled.

13. A number is composed of two digits, the first of which exceeds the second by unity, and the number itself falls short of the sum of the squares of its digits by 26. Find the number.

14. A number consists of two digits, whose sum is 8; another number is obtained by reversing the digits. If the product of the two is 1855, find the numbers.

15. A vessel can be filled by two pipes, running together, in $22\frac{1}{2}$ minutes; the larger pipe can fill the vessel in 24 minutes less than the smaller one. Find the time taken by each.

Solution. Let $x =$ the *number* of minutes it takes the larger pipe.

Then, $x + 24 =$ the *number* of minutes it takes the smaller pipe.

$\frac{1}{x} =$ the *part* filled by the larger pipe in *one* minute,

and $\frac{1}{x+24} =$ the *part* filled by the smaller pipe in *one* minute.

Therefore, $\frac{1}{x} + \frac{1}{x+24} = \frac{1}{22\frac{1}{2}}.$

The solution of which gives, $x = 36$, or -15.

One pipe will fill it in 36 minutes, and the other in 1 hour.

16. A vessel can be filled by two pipes, running together, in m minutes; the larger pipe can fill the vessel in n minutes less than the smaller one. Find the time taken by each. Solve when $m = 56$ and $n = 66$.

17. B can do some work in 4 hours less time than A can do it, and together they can do it in $3\frac{3}{4}$ hours. How many hours will it take each alone to do it?

18. A boat's crew row 7 miles down a river and back in 1 hour and 45 minutes. If the current of the river is 3 miles per hour, find the rate of rowing in still water.

19. A boat's crew row a miles down a river and back. They can row m miles an hour in still water. It took n hours longer to row against the current than the time to row with it. Find the rate of the current. Solve when $a = 5$, $m = 6$, and $n = 2$.

20. A uniform iron bar weighs m pounds. If it was a feet longer each foot would weigh n pounds less. Find the length and weight per foot. Solve when $m = 36$, $a = 1$, and $n = \frac{1}{2}$.

21. A and B agree to do some work in a certain number of days. A lost m days of the time and received n dollars. B lost a days and received c dollars. Had A lost a days and B m days, the amounts received would have been equal. Find the number of days agreed on and the daily wages of each. Solve when $m = 4$, $n = 18.75$, $a = 7$, and $c = 12$.

22. A person sold goods for m dollars, and gained as much per cent as the goods cost him. Find the cost of the goods. Solve when $m = 144$.

23. By selling goods for m dollars, I lose as much per cent as the goods cost me. Find the cost of the goods. Solve when $m = 24$.

CHAPTER XXII.

EQUATIONS WHICH MAY BE SOLVED AS QUADRATICS.

139. In the equation $m(y^3-y)^4+n(y^3-y)^2+a=0$, suppose $(y^3-y)^2=x$, then $mx^2+nx+a=0$. Similarly, $y^{\frac{3}{2}}-3y^{\frac{3}{4}}-9=0$ may be changed to the form $x^2-3x-9=0$.

Hence, an equation is in the quadratic form when the unknown number is found in two terms affected with two exponents, one of which is twice the other; as, $x^6+5x^3-8=0$.

The general form for an equation in the **quadratic form** is,

$$ax^{2n}+bx^n+c=0;$$

where a, b, c, and n represent any numbers whatever, positive or negative, integral or fractional.

Example 1. Solve $x^4-13x^2+36=0$.

Process. Factor, $(x+2)(x-2)(x+3)(x-3)=0$.
Hence, $x+2=0$, $x-2=0$, $x+3=0$, and $x-3=0$.
Therefore, $x=\pm 2$, or ± 3.

Example 2. Solve $8x^{-\frac{6}{5}}-15x^{-\frac{3}{5}}-2=0$.

Process. Factor, $(x^{-\frac{3}{5}}-2)(8x^{-\frac{3}{5}}+1)=0$.
$\therefore\quad x^{-\frac{3}{5}}-2=0$, or $x^{\frac{3}{5}}=\frac{1}{2}$. $\quad x=(\frac{1}{2})^{\frac{5}{3}}=\frac{1}{4}\sqrt[3]{2}$.
Also, $8x^{-\frac{3}{5}}+1=0$, or $x^{\frac{3}{5}}=-8$. $\quad x=(-8)^{\frac{5}{3}}=-32$.

Example 3. Solve $3x+x^{\frac{1}{2}}-2=0$.

Process. Solve for $x^{\frac{1}{2}}$. Thus,
Multiply by 12 and transpose, $36x+12x^{\frac{1}{2}}=24$.
Complete the square, $36x+12x^{\frac{1}{2}}+1=25$.
Extract the square root, $6x^{\frac{1}{2}}+1=\pm 5$.
Therefore, $x^{\frac{1}{2}}=\frac{2}{3}$, or -1.
Square each member, $x=\frac{4}{9}$, or 1.

Example 4. Solve $2\sqrt[5]{x^{-2}} + 3\sqrt[5]{x^{-4}} - 56 = 0$.

Process. Since $\sqrt[5]{x^{-2}}$ is the same as $x^{-\frac{2}{5}}$, and $\sqrt[5]{x^{-4}}$ is the same as $x^{-\frac{4}{5}}$, this equation is in the quadratic form. Transpose and multiply by 12,

$$36x^{-\frac{4}{5}} + 24x^{-\frac{2}{5}} = 672.$$

Complete the square, $36x^{-\frac{4}{5}} + 24x^{-\frac{2}{5}} + 4 = 676.$

Extract the square root, $6x^{-\frac{2}{5}} + 2 = \pm 26.$

$$x^{-\frac{2}{5}} = 4, \text{ or } -\tfrac{14}{3}.$$

Therefore, $x^{\frac{2}{5}} = \frac{1}{4}$, or $-\frac{3}{14}$.

Extract the square root, $x^{\frac{1}{5}} = \pm\frac{1}{2}$, or $\pm\sqrt{-\frac{3}{14}}$.

Raise to the 5th power,* $x = \pm\frac{1}{32}$, or $\pm\sqrt{(-\frac{3}{14})^5}$.

Notes: 1. When the roots cannot all be obtained by completing the square, the method by factoring should be used. Thus, in solving $x^6 + 7x^3 - 8 = 0$, by completing the square, we find but two values for x, $x = 1$, or -2. Factoring the first member, we have $(x+2)(x^2-2x+4)(x-1)(x^2+x+1) = 0$. Hence, $x+2=0$, $x^2-2x+4=0$, $x-1=0$, and $x^2+x+1=0$. Solving these equations, $x = -2$, $1 \pm \sqrt{-3}$, 1, and $\dfrac{-1 \pm \sqrt{-3}}{2}$.

2. * In solving equations of the form $x^{\frac{m}{n}} = a$, first extract the mth root, and then raise to the nth power. In practice this is the same as affecting the equation by the exponent $\frac{n}{m}$. Thus, $x = a^{\frac{n}{m}}$.

Example 5. Solve $ax^{2n} + bx^n = -c$.

Process. Multiply by $4a$,

$$4a^2x^{2n} + 4abx^n = -4ac.$$

Complete the square,

$$4a^2x^{2n} + 4abx^n + b^2 = -4ac + b^2.$$

Extract the square root, $2ax^n + b = \pm\sqrt{b^2 - 4ac}.$

Transpose b and divide by $2a$, $x^n = \dfrac{\pm\sqrt{b^2-4ac} - b}{2a}.$

Extract the **nth** root, $x = \left[\dfrac{\pm\sqrt{b^2-4ac} - b}{2a}\right]^{\frac{1}{n}}$ (i)

Example 6. Solve $4x^4 - 37x^2 + 9 = 0$.

Process. Here, $a = 4$, $b = -37$, $c = 9$, and $n = 2$.

Substitute these values in (i), $x = \left[\dfrac{\pm\sqrt{(-37)^2 - 4\times 4\times 9} - (-37)}{2\times 4}\right]^{\frac{1}{2}}$

$$= \left[\frac{\pm 35 + 37}{8}\right]^{\frac{1}{2}}$$

$$= \pm 3, \text{ or } \pm \tfrac{1}{2}.$$

Exercise 120.

Solve the following equations:

1. $x^4 - 14x^2 = -40$; $x^{10} + 31x^5 = 32$; $x^6 - 7x^3 = 8$.

2. $x^3(19 + x^3) = 216$; $x^2 + \dfrac{a^2b^2}{x^2} = a^2 + b^2$.

3. $16\left(x^2 + \dfrac{1}{x^2}\right) = 257$; $x^3 + 14x^{\frac{3}{2}} = 1107$.

4. $5x^{\frac{1}{2}} + \sqrt[4]{x} = 22$; $\sqrt[3]{x} + \dfrac{5}{2\sqrt[3]{x}} = 3\tfrac{1}{4}$.

5. $x^{\frac{4}{3}} + 7x^{\frac{2}{3}} = 44$; $3x^3 + 42x^{\frac{3}{2}} = 3321$.

6. $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$; $3\sqrt{x^3} - 4\sqrt[4]{x^3} = 7$.

7. $2\sqrt{x} + \dfrac{2}{\sqrt{x}} = 5$; $12x^{-\frac{2}{3}} + \frac{5}{3} = \dfrac{2}{x^{\frac{2}{3}}} + \frac{25}{9}$.

8. $3x^{\frac{3}{2}} - x^{-\frac{3}{2}} + 2 = 0$; $2x^{-5} + 61x^{-\frac{5}{2}} - 96 = 0$.

9. $x^{-1} + ax^{-\frac{1}{2}} = 2a^2$; $x^{-2} - 2x^{-1} = 8$; $x^{-1} + \sqrt{x^{-1}} = 6$.

10. $x^{4n} - \frac{5}{3}x^{2n} - \frac{25}{12} = 0$; $3x^{\frac{4}{3}n} + 4x^{\frac{2}{3}n} = 4$.

11. $x^{\frac{m}{n}} + 13x^{\frac{m}{2n}} = 14$; $3x^{-\frac{3}{2m}} + 26x^{-\frac{3}{4m}} = -16$.

140. Equations may frequently be put in the quadratic form by **grouping the terms** containing the unknown number, so that the exponent of one group shall be twice the exponent of the other group, and then solved for the polynomial. Thus,

EXAMPLE 1. Solve $x - 3x^{\frac{1}{2}} - 4\sqrt{x - 3x^{\frac{1}{2}} - 1} = -2$.

The equation may be put in the *quadratic form* if we regard $\sqrt{x - 3x^{\frac{1}{2}} - 1}$ as the unknown number. Thus,

Process. Add -1 to each member,

$$x - 3x^{\frac{1}{2}} - 1 + 4\sqrt{x - 3x^{\frac{1}{2}} - 1} = -3.$$

Put $\sqrt{x - 3x^{\frac{1}{2}} - 1} = y$, $\quad y^2 + 4y = -3.$

Therefore, $\quad y = 3$, or 1.

Hence, $\quad \sqrt{x - 3x^{\frac{1}{2}} - 1} = 3$, or 1.

Squaring, $\quad x - 3x^{\frac{1}{2}} - 1 = 9$, or 1.

Complete the square, $\quad x - 3x^{\frac{1}{2}} + \frac{9}{4} = \frac{49}{4}$, or $\frac{17}{4}$.

Solving these equations for the values of x, we find $x = 25$, or 4, and $x = \dfrac{13 \pm 3\sqrt{17}}{2}$.

Note 1. In solving equations of this form we must group the terms so that the expression outside of the radical, in the first member, is the same or a multiple of the expression under the radical sign.

EXAMPLE 2. Solve $x^4 - 6ax^3 + 7a^2x^2 + 6a^3x = 24a^4$.

Process. Add $2a^2x^2$, $\quad x^4 - 6ax^3 + 9a^2x^2 + 6a^3x = 24a^4 + 2a^2x^2$.

Transpose $2a^2x^2$, $x^4 - 6ax^3 + 9a^2x^2 + 6a^3x - 2a^2x^2 = 24a^4$.

Group and factor the terms,

$$(x^2 - 3ax)^2 - 2a^2(x^2 - 3ax) = 24a^4.$$

Regard $x^2 - 3ax$ as the unknown number, and complete the square,

$$(x^2 - 3ax)^2 - 2a^2(x^2 - 3ax) + a^4 = 25a^4.$$

Extract the square root, $\quad (x^2 - 3ax) - a^2 = \pm 5a^2$.

Therefore, $\quad x^2 - 3ax = 6a^2$, or $-4a^2$.

Complete the square and solve, $\quad x = \dfrac{a}{2}(3 \pm \sqrt{33})$,

$$x = \frac{a}{2}(3 \pm \sqrt{-7}).$$

Note 2. Form a perfect square with x^4 and $-6ax^3$. The third term of the square is the square of the quotient obtained by dividing $6ax^3$ by twice the square root of x^4.

Example 3. Solve $x^2+4x-4x^{-1}+x^{-2}=\frac{7}{9}$.

Process. Use positive exponents, rearrange terms, and factor,

$$x^2+\frac{1}{x^2}+4\left(x-\frac{1}{x}\right)=\tfrac{7}{9}.$$

Regard $x-\frac{1}{x}$ as the unknown number, and subtract 2 from both sides,

$$x^2-2+\frac{1}{x^2}+4\left(x-\frac{1}{x}\right)=-\tfrac{11}{9}.$$

Factor, and complete the square,

$$\left(x-\frac{1}{x}\right)^2+4\left(x-\frac{1}{x}\right)+4=\tfrac{25}{9}.$$

Extract the square root, $\quad x-\frac{1}{x}+2=\pm\frac{5}{3}.$

Therefore, $\quad x-\frac{1}{x}=-\frac{1}{3}, \text{ or } -\frac{11}{3}.$

Free from fractions, $\quad x^2-1=-\frac{1}{3}x, \text{ or } -\frac{11}{3}x.$

Complete the square and solve, $\quad x=\frac{1}{6}\left(-1\pm\sqrt{37}\right),$

$$x=\tfrac{1}{6}\left(-11\pm\sqrt{157}\right).$$

Note 3. Form a perfect square with x^2 for the first term and $\frac{1}{x^2}$ for the third. The middle term will be twice the product of their square roots taken with a negative sign.

A **Biquadratic Equation** is an equation of the fourth degree. Biquadratic means twice squared, and hence the fourth power.

If a biquadratic is in the form,

$$x^4+2mx^3+(m^2+2n)x^2+2mnx=a \qquad \text{(ii)}$$

the first member becomes a perfect square by

Adding n^2, *or the square of the quotient obtained by dividing the coefficient of* x *by the coefficient of* x^3.

Thus, extracting the square root of the first member,

$$
\begin{array}{r|l|l}
 & x^4 + 2\,m\,x^3 + (m^2 + 2\,n)\,x^2 + 2\,m\,n\,x & x^2 + m\,x + n \\
 & x^4 & \\ \hline
2\,x^2 + m\,x & 2\,m\,x^3 + (m^2 + 2\,n)\,x^2 + 2\,m\,n\,x & \\
 & 2\,m\,x^3 + (m^2 \qquad\quad)\,x^2 & \\ \hline
2\,x^2 + 2\,m\,x + n & +(\qquad\quad 2\,n)\,x^2 + 2\,m\,n\,x & \\
 & +(\qquad\quad 2\,n)\,x^2 + 2\,m\,n\,x + n^2 & \\ \hline
 & \qquad\qquad\qquad\qquad\qquad - n^2. &
\end{array}
$$

Hence, the equation may be written,

$$(x^2 + m\,x + n)^2 - n^2 = a, \text{ or } (x^2 + m\,x + n)^2 = a + n^2 \qquad \text{(iii)}$$

EXAMPLE 4. Solve $x^4 - 10\,x^3 + 35\,x^2 - 50\,x = 11$.

Process. Here, $2\,m = -10$, $2\,m\,n = -50$. $\therefore\ m = -5$ and $n = 5$.
Since $m^2 + 2\,n = 35$, the equation has the form of (ii).
Add 25; or put $m = -5$, $n = 5$, and $a = 11$ in (iii),

$$(x^2 - 5\,x + 5)^2 = 36.$$

Extract the square root, $\quad x^2 - 5\,x + 5 = \pm 6.$

Therefore, $\quad x^2 - 5\,x = 1, \text{ or } -11.$

Complete the square and solve, $\quad x = \dfrac{5 \pm \sqrt{29}}{2},$

$$x = \frac{5 \pm \sqrt{-19}}{2}.$$

Note 4. After adding the value for n^2 the first member may be factored by substituting the values for m and n in (iii).

Exercise 121.

Solve the following equations:

1. $(x^2 + x - 2)^2 - 13\,(x^2 + x - 2) + 36 = 1.$

2. $x^2 + 2\,x + 6\sqrt{x^2 + 2\,x + 5} = 11.$

3. $x^2 + 24 = 12\sqrt{x^2 + 4}$; $\quad 2\,x + 17 = 9\sqrt{2\,x - 1}.$

4. $x^2 - x + 5\,(2\,x^2 - 5\,x + 6)^{\frac{1}{2}} = \frac{1}{2}\,(3\,x + 33).$

5. $\dfrac{(x^2+x+6)^{\frac{1}{2}}}{3} = \dfrac{20 - \frac{4}{3}(x^2+x+6)^{\frac{1}{2}}}{(x^2+x+6)^{\frac{1}{2}}}$.

6. $\left(x - \dfrac{1}{x}\right)^2 + \dfrac{5}{6}\left(x - \dfrac{1}{x}\right) = 1$.

7. $\left(x + \dfrac{1}{x}\right)^2 + 4\left(x + \dfrac{1}{x}\right) = 12$.

8. $(x^2 - 5x)^2 - 8(x^2 - 5x) = 48$.

9. $9x - 3x^2 + 4(x^2 - 3x + 5)^{\frac{1}{2}} = 11$.

10. $\left(2 - \dfrac{1}{x}\right)^2 - \dfrac{2}{x}\left(2 - \dfrac{1}{x}\right) = \dfrac{15}{x^2}$.

11. $(3x^2 - 10x + 5)^2 - 8(3x^2 - 10x + 5) = 9$.

12. $\left(\dfrac{x+1}{x-1}\right)^2 + 7\left(\dfrac{x+1}{x-1}\right) = 12$.

13. $x^4 + 6x^3 + 5x^2 - 12x = -12$.

14. $x^4 - 6x^3 - 29x^2 + 114x = 80$.

15. $x^4 + 2x^3 - 25x^2 - 26x + 120 = 0$.

16. $x^4 - 8x^3 + 10x^2 + 24x + 5 = 0$.

17. $x^4 + 3x^3 + 2x^2 - \frac{3}{8}x = \frac{5}{4}$.

18. $(x^3 - 16)^{\frac{2}{3}} - 3(x^3 - 16)^{\frac{1}{3}} = 4$.

19. $x^2 + x^{-2} + x - x^{-1} = 4$; $x^2 + 3x - 3x^{-1} + x^{-2} = \frac{7}{36}$.

141. Equations Containing Radicals may be Solved. Thus,

EXAMPLE 1. Solve $x - \sqrt[3]{x^3 + 2x + 12} + 2 = 0$.

Process. Transpose, $$x + 2 = \sqrt[3]{x^3 + 2x + 12}.$$
Raise both members to the third power,
$$x^3 + 6x^2 + 12x + 8 = x^3 + 2x + 12.$$
Transpose and simplify, $$3x^2 + 5x - 2 = 0.$$
Factor and solve, $$x = \tfrac{1}{3}, \text{ or } -2.$$
Verify by putting these numbers for x in the original equation.

Process. $x = \frac{1}{3}$.
$$\tfrac{1}{3} - \sqrt[3]{\tfrac{1}{27} + \tfrac{2}{3} + 12} + 2 = 0,$$
$$\tfrac{1}{3} - \tfrac{7}{3} + 2 = 0,$$
$$0 = 0.$$

$x = -2$.
$$-2 - \sqrt[3]{-8 - 4 + 12} + 2 = 0,$$
$$-2 - 0 + 2 = 0,$$
$$0 = 0,$$

EXAMPLE 2. Solve $\dfrac{1}{\sqrt{x^2+1}} + \dfrac{1}{\sqrt{x^2-1}} = \dfrac{1}{\sqrt{x^4-1}}$.

Process. Multiply by $\sqrt{x^4 - 1}$,
$$\sqrt{x^2 - 1} + \sqrt{x^2 + 1} = 1.$$
Square, $$x^2 - 1 + 2\sqrt{x^4 - 1} + x^2 + 1 = 1.$$
Transpose and simplify, $$2\sqrt{x^4 - 1} = 1 - 2x^2.$$
Square, $$4x^4 - 4 = 1 - 4x^2 + 4x^4.$$
Simplify, $$x^2 = \tfrac{5}{4}.$$
Extract the square root, $$x = \pm \tfrac{1}{2}\sqrt{5}.$$

Exercise 122.

Solve the following equations:

1. $3\sqrt{x+6} + 2 = x + \sqrt{x+6}$; $x + \sqrt{x+2} = 10$.

2. $x + 16 - 7\sqrt{x+16} = 10 - 4\sqrt{x+16}$.

3. $2x + \sqrt{4x+8} = \frac{7}{2}$; $\sqrt{4x+17} + \sqrt{x+1} = 4$.

4. $2\sqrt{3x+7}=9-\sqrt{2x-3}$; $\dfrac{\sqrt{4x}+2}{4\sqrt{x}}=\dfrac{4-\sqrt{x}}{\sqrt{x}}$.

5. $\sqrt{x+5}=\dfrac{12}{\sqrt{x+12}}$; $\dfrac{5x-9}{\sqrt{5x}+3}=1+\dfrac{\sqrt{5x}-3}{2}$.

6. $\sqrt{x^3}-2\sqrt{x}=x$; $\sqrt[3]{64+2x^2-8x}=\dfrac{4+x}{\sqrt[3]{4+x}}$.

7. $\sqrt{\dfrac{y}{4}+3}+\sqrt{\dfrac{y}{4}-3}=\sqrt{\tfrac{2}{3}y}$; $\dfrac{3x-1}{\sqrt{3x}+1}=1+\dfrac{\sqrt{3x}-1}{2}$.

8. $\dfrac{2}{x+\sqrt{2-x^2}}+\dfrac{2}{x-\sqrt{2-x^2}}=x$.

9. $\dfrac{\sqrt{7y^2+4}+2\sqrt{3y-1}}{\sqrt{7y^2+4}-2\sqrt{3y-1}}=7$; $\dfrac{m-\sqrt{m^2-y^2}}{m+\sqrt{m^2-y^2}}=n$.

10. $\sqrt[3]{a^2+2x^2-2ax}=\dfrac{a+x}{\sqrt[3]{a+x}}$; $\sqrt{6x-x^2}=\dfrac{1+x^2}{\sqrt{x}}$.

11. $\dfrac{ax-b^2}{\sqrt{ax}+b}=\dfrac{\sqrt{ax}+b}{c}$; $\dfrac{\sqrt{x}+9}{\sqrt{x}}=\dfrac{3\sqrt{x}-3.8}{9-\sqrt{x}}$.

12. $\sqrt{a+x}+\sqrt{a-x}=\dfrac{12a}{5\sqrt{a+x}}$; $\dfrac{6+3\sqrt{x}}{4+\sqrt{x}}=\dfrac{4}{\sqrt{x}}$.

13. $\sqrt{a+y}-\sqrt{y-a}=\sqrt{2y}$; $2x+3\sqrt{x}=27$.

14. $\dfrac{x+\sqrt{x}}{x-\sqrt{x}}=\dfrac{x^2-x}{4}$; $x=\dfrac{12+8\sqrt{x}}{x-5}$.

15. $x-3=\dfrac{3+4\sqrt{x}}{x}$; $\dfrac{x}{4}=\dfrac{\sqrt{x}-12}{x-18}$.

16. $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}$; $\sqrt{x-a}+\sqrt{x-a}=\sqrt{x}$.

THEORY OF QUADRATIC EQUATIONS.

142. Representing the roots of $mx^2 + nx = -a$ by r and r_1, we have (Art. 134),

$$r = \frac{-n + \sqrt{n^2 - 4am}}{2m},$$

$$r_1 = \frac{-n - \sqrt{n^2 - 4am}}{2m}.$$

Adding, $$r + r_1 = -\frac{n}{m} \qquad \text{(i)}$$

Multiplying, $$rr_1 = \frac{a}{m} \qquad \text{(ii)}$$

Hence, if a quadratic appears in the form, $mx^2 + nx = -a$,

I. *The sum of the roots is equal to the quotient, with its sign changed, obtained by dividing the coefficient of* x *by the coefficient of* x^2.

II. *The product of the roots is equal to the second member, with its sign changed, divided by the coefficient of* x^2.

By means of (i) and (ii) the original equation becomes,

$$mx^2 - m(r + r_1)x + mrr_1 = 0 \qquad (1)$$

Factor, $$m(x - r)(x - r_1) = 0 \qquad (2)$$

If $m = 1$, $$x^2 + nx = -a$$

$$x^2 - (r + r_1)x + rr_1 = 0 \qquad \text{(iii)}$$

$$(x - r)(x - r_1) = 0 \qquad (3)$$

If the roots of a quadratic equation be given, by means of (iii) we can readily form the equation.

EXAMPLE 1. Form the equation whose roots are $\frac{1}{2}, -\frac{1}{4}$.

Process. Here, $r = \frac{1}{2}$ and $r_1 = -\frac{1}{4}$.
Substitute these values in (iii), $x^2 - (\frac{1}{2} - \frac{1}{4})x + (\frac{1}{2})(-\frac{1}{4}) = 0$.
Simplify, $8x^2 - 2x - 1 = 0$.

EXAMPLE 2. Find the sum and the product of the roots of $8x^2 + 3x - 5 = 0$.

Process. Here, $m = 8$, $n = 3$, and $a = -5$.
Substitute in (i) and (ii), $r + r_1 = -\frac{3}{8}$ and $rr_1 = -\frac{5}{8}$.

Exercise 123.

Find the sum and product of the roots of:

1. $x^2 + 8x = 9$; $12x^2 - 187x + 588 = 0$.

2. $20x^{-2} = 5 - 5x^{-1}$; $x^2 - 6x + 9 = 9x$.

3. $3x^2 + 5 = 0$; $x^2 + a^2 = ax$; $x^2 - 15x = 3$.

4. $x^2 - \dfrac{2mn^{\frac{1}{2}}x}{m-n} = -\dfrac{mn}{m-n}$; $x^2 - 2bx - a^2 + b^2 = 0$.

Form the equations whose roots are:

5. $7, -3$; $\frac{2}{3}, -\frac{3}{2}$; $5, -3$; $\pm\sqrt{-3}$; $2-\sqrt{3}, 2+\sqrt{3}$.

6. $0, -5$; $7+2\sqrt{5}, 7-2\sqrt{5}$; $1+\sqrt{2}, 1-\sqrt{2}$.

7. $m(m+1), 1-m$; $\dfrac{m}{n}, -\dfrac{n}{m}$; $\dfrac{(a+b)^2}{a-b}, b-a$.

8. $-m+2\sqrt{2n}, -m-2\sqrt{2n}$; $\dfrac{a\pm\sqrt{b}}{2}, \dfrac{\sqrt{a}\pm\sqrt{b}}{4}$.

143. A **Root** is said to be a **Surd** when it can be found only approximately; as, $x = \pm\sqrt{5}$.

Real Roots are values of the unknown numbers that can be found either exactly or approximately.

Imaginary Roots are values of the unknown numbers that cannot be found exactly or approximately; as,

$$x = \pm\sqrt{-5}.$$

Character of Roots. For brevity, represent the roots of the equation $mx^2 + nx + a = 0$ by r and r_1, then,

$$r = \frac{-n + \sqrt{n^2 - 4am}}{2m},$$

$$r_1 = \frac{-n - \sqrt{n^2 - 4am}}{2m}.$$

It is seen that the two roots have the same expression, $\sqrt{n^2-4am}$.

If n^2 is *greater* than $4am$, n^2-4am will be *positive*, and $\sqrt{n^2-4am}$ can be found exactly or approximately.

If n is *positive*, r_1 is numerically greater than r; if n is *negative*, r is numerically greater than r_1. Hence,

I. **Condition for Real and Different Roots.** n^2-4am, *positive.*

Illustration. $$3x^2-2x+\tfrac{1}{8}=0.$$

Here, $$m=3,\ n=-2, \text{ and } a=\tfrac{1}{8}.$$

$$n^2-4am=(-2)^2-4\times\tfrac{1}{8}\times 3=4-\tfrac{3}{2}=\tfrac{5}{2}.$$

Therefore, the roots are real and different.

Evidently both roots will be *rational* or both *surds* according as n^2-4am is, or is not, a perfect square. Hence,

II. **Condition for a Rational or a Surd Root.** n^2-4am, *a square number;* or, $\sqrt{n^2-4am}$, *a surd.*

Illustrations. (1) $x^2-3x-4=0$; (2) $8x^2+5x-\frac{1}{4}=0$.

(1) Here, $$m=1,\ n=-3, \text{ and } a=-4.$$

$$n^2-4am=(-3)^2-4\times-4\times 1=9+16=25.$$

Therefore, the roots are real and rational, and different.

(2) Here, $m=8$, $n=5$, and $a=-\frac{1}{4}$.

$$\sqrt{n^2-4am}=\sqrt{25+8}=\sqrt{33}.$$

Therefore, the roots are real and surds, and different.

If n^2 is *less* than $4am$, n^2-4am will be *negative*, and $\sqrt{n^2-4am}$ will represent the *even* root of a *negative* number. Hence,

III. **Condition for Imaginary Roots.** n^2-4am, *negative.*

Illustration. $$2x^2-3x+2=0.$$

Here, $$m=2,\ n=-3, \text{ and } a=2.$$

$$n^2-4am=(-3)^2-4\times 2\times 2=9-16=-7.$$

Therefore, the roots are both imaginary.

If $n^2=4am$, $n^2-4am=0$, and the roots will be *real* and *equal*, and have the same sign, but opposite to that of n. Hence,

IV. Condition for Equal Roots. $n^2 - 4am = 0$.

Illustration. $$4x^2 - 12x + 9 = 0.$$

Here, $$m = 4,\ n = -12, \text{ and } a = 9.$$
$$n^2 - 4am = 144 - 144 = 0.$$

Therefore, the roots are real and equal.

If am is *positive*, for *real* roots, $n^2 - 4am$ will be *positive* and *less* than n^2, since $\sqrt{n^2 - 4am}$ will be *less* than n.

If am is *negative*, $\sqrt{n^2 - 4am}$ will be *greater* than n, since $n^2 - 4am$ will be greater than n^2. Hence,

V. Condition for Signs. *If* am *is positive, real roots have the same sign but opposite to that of* n. *If* am *is negative, the roots have opposite signs.*

Illustrations. (1) $2x^2 - 10x + 12 = 0$; (2) $3x^2 - 5x - 3\frac{1}{12} = 0$.

(1) Here, $$m = 2,\ n = -10, \text{ and } a = 12.$$
$$n^2 - 4am = 100 - 96 = 4.$$

Therefore, the roots are rational and positive, and different.

(2) Here, $$m = 3,\ n = -5, \text{ and } a = -3\tfrac{1}{12}.$$
$$n^2 - 4am = 25 + 47 = 72.$$

The roots are surds and have opposite signs, and different.

Exercise 124.

Determine by inspection the character of the roots of:

1. $5x^2 - x = 3$; $7x^2 + 2x = -\frac{1}{7}$; $11x^2 - x = -4\frac{1}{4}$.

2. $4x^2 + 52x = 87$; $3x^2 + 4x + 4 = 0$.

3. $5 - 1\frac{1}{2}x - 9x^2 = 0$; $9x = 3 + 4\frac{1}{2}x^2$.

4. $10x + 8\frac{1}{3} = -3x^2$; $\frac{3}{2}x^2 - \frac{3}{4}x + \frac{1}{2} = 0$.

5. $3x^2 - 2x + 3 = 0$; $4x^2 - 3x - 5 = 0$.

6. $x^2+\frac{3}{2}x=-\frac{9}{16}$; $2x^2-18=0$; $x^2-14x+45=0$.

7. $28=3x+x^2$; $\frac{1}{7}x^{-2}-\frac{2}{3}x^{-1}=5$; $x^2-x-1=0$.

8. $6x^2+5x-21=0$; $13x^2+56x-605=0$.

9. $9x^2-30x+41=0$; $40x^2-100x-360=0$.

Query. How many roots can a quadratic equation have? Why?

Miscellaneous Exercise 125.

Solve the following equations:

1. $x^{\frac{4}{3}}+7x^{\frac{2}{3}}=44$; $x^{-2}-2x^{-1}=8$; $3x^{\frac{1}{2n}}-x^{\frac{1}{n}}-2=0$.

2. $\sqrt{\dfrac{x}{1-x}}+\sqrt{\dfrac{1-x}{x}}=2\frac{1}{6}$; $1+8x^{\frac{6}{5}}+9\sqrt[5]{x^3}=0$.

3. $\dfrac{30}{\sqrt{2x}}-2\sqrt{2x}=59$; $\dfrac{15x-5}{1+5\sqrt{x}}+\dfrac{2}{\sqrt{x}}=3\sqrt{x}$.

4. $x^{4n}-2x^{3n}+x^n=6$; $x^4-2x^3+x=a$.

5. $x^4+\frac{13}{3}x^3-39x=81$; $x^4-2x^3+x=380$.

6. $108x^4=180x^3-20x-51x^2+7$.

7. $x^4-10x^{10}+35x^2-50x=-24$.

8. $(x-a)^{\frac{5}{3}}+2\sqrt{n}\,(x-a)^{\frac{5}{6}}-3n=0$.

9. $x^{\frac{4}{3}}+4x^{\frac{2}{3}}+x^{-\frac{4}{3}}+4x^{-\frac{2}{3}}=-\frac{7}{4}$.

10. $\dfrac{\sqrt{y+2a}-\sqrt{y-2a}}{\sqrt{y-2a}+\sqrt{y+2a}}=\dfrac{y}{2a}$; $\dfrac{x+\sqrt{x}}{x-\sqrt{x}}=\dfrac{x^2-x}{4}$.

11. $3x^n\sqrt[3]{x^n}-\dfrac{4x^n}{\sqrt[3]{x^n}}=4$; $\sqrt{6x+1}+\sqrt{x+4+\sqrt{6x+1}}=2$.

12. $x\sqrt{5}+\sqrt{2x+2}=\sqrt{x+2}$; $x-1=2+\dfrac{2}{\sqrt{x}}$.

13. $2\sqrt{x}+2x^{-\frac{1}{2}}=5$; $6\sqrt{x}=5x^{-\frac{1}{2}}-13$.

14. $x^{\frac{1}{2}}+2m^2x^{-\frac{1}{2}}=3m$; $x^{-\frac{1}{3}}+2=\dfrac{x^{-1}+8}{x^{-\frac{2}{3}}+5}$.

15. $\sqrt{x+\sqrt{2x-1}}-\sqrt{x-\sqrt{2x-1}}=\dfrac{3}{5}\sqrt{\dfrac{10x}{x+\sqrt{2x-1}}}$.

16. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}-\dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}=8x\sqrt{x^2-3x+2}$.

17. $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x}=b$; $x^3+x\sqrt{x}-72=0$.

18. State the conditions that will make the roots of $x^2+Ax+B=0$: (i) surds; (ii) real; (iii) imaginary; (iv) equal; (v) have same signs; (vi) have opposite signs; (vii) equal in value but opposite in sign.

19. Find a number such that if its nth root be increased by one half of its $\frac{n}{2}$th root, the sum shall be a. Solve when $n=2$ and $a=12$.

20. Find a number such that if its nth power be diminished by the $\frac{2}{n}$th root of the $\frac{a}{c}$th part of it, the remainder shall be m. Solve when $m=144$, $n=2$, $a=27$, and $c=5$.

CHAPTER XXIII.

SIMULTANEOUS QUADRATIC EQUATIONS.

144. Only certain forms of quadratic equations involving two unknown numbers can be solved. Thus,

Example 1. Solve the equations: $\begin{cases} 2x + y = 10 & (1) \\ 2x^2 - xy + 3y^2 = 54 & (2) \end{cases}$

Process. From (1), $x = \dfrac{10 - y}{2}$ (3)

Substitute in (2), $2\left(\dfrac{10-y}{2}\right)^2 - \left(\dfrac{10-y}{2}\right)y + 3y^2 = 54.$

Simplify and factor, $(y - 4)(4y + 1) = 0.$

Therefore, $y = 4$, or $-\frac{1}{4}$.

Substitute in (3), $x = 3$, or $5\frac{1}{8}$. Hence,

When one of the Equations is of the First Degree. Solve by substitution.

The **Degree** of a term is the number of *literal factors* involved, and is always equal to the *sum* of their exponents.

Each literal factor is called a **Dimension.**

Thus, $3xy$ is of the *second degree*, and has *two dimensions*.
$5x^3y^2$ is of the *fifth degree*, and has *five* dimensions.

Example 2. Solve the equations: $\begin{cases} 183xy + 72x + 36y = 88 & (1) \\ 177xy + 60x + 36y = 80 & (2) \end{cases}$

Process. From (2), $x = \dfrac{80 - 36y}{177y + 60}$ (3)

Substitute in (1),

$$\frac{183(80y - 36y^2)}{177y + 60} + \frac{72(80 - 36y)}{177y + 60} + 36y = 88.$$

Simplify, $9y^2 + 57y - 20 = 0.$

Complete the square and solve, $y = \frac{1}{3}$, or $-6\frac{2}{3}$.

Substitute in (3), $x = \frac{4}{7}$, or $-\frac{2}{7}$.

EXAMPLE 3. Solve the equations: $\begin{cases} 6x^2 - x - 3y = 5 & (1) \\ x^2 + x - y = 1 & (2) \end{cases}$

Process. From (1), $$y = \frac{6x^2 - x - 5}{3} \qquad (3)$$

Substitute in (2) and simplify,
$$3x^2 - 4x - 2 = 0.$$

Complete the square and solve, $$x = \frac{2 \pm \sqrt{10}}{3}.$$

Substitute in (3), $$y = \frac{11 \pm 7\sqrt{10}}{9}.$$ Hence,

When each Equation Contains only one Second Degree Term, and that Term Consists of the Same Product or Square of the Unknown Numbers. Solve by substitution.

Exercise 126.

Solve the following equations:

1. $\begin{cases} 5xy = 50. \\ \dfrac{4x + 2y}{3} = 6. \end{cases}$

2. $\begin{cases} 3xy + 6x - 2y = 4. \\ 4xy - x + 4y = 1. \end{cases}$

3. $\begin{cases} x + xy = 24. \\ xy + y = 21. \end{cases}$

4. $\begin{cases} 2y^2 + y = 28. \\ y^2 + 3x - 4y = 18. \end{cases}$

5. $\begin{cases} 15 + y = x. \\ xy = 2y^3. \end{cases}$

6. $\begin{cases} xy + 6x + 7y = 66. \\ 3xy + 2x + 5y = 70. \end{cases}$

7. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{7}{12}. \\ xy = 12. \end{cases}$

8. $\begin{cases} 2x + y = 22. \\ y^2 + \dfrac{xy}{2} = 60. \end{cases}$

9. $\begin{cases} x - y = 5. \\ xy = 126. \end{cases}$

10. $\begin{cases} x^3 - y^3 = 218. \\ x - y = 2. \end{cases}$

11. $\begin{cases} x - y = 4. \\ x^2 + y^2 = 106. \end{cases}$

12. $\begin{cases} x + 3y = 16. \\ 3x^2 + 2xy - y^2 = -12. \end{cases}$

13. $\begin{cases} x^2 + y^2 = 185. \\ x - y = 3. \end{cases}$

14. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 2. \\ x + y = 2. \end{cases}$

15. $\begin{cases} x + y = 9. \\ x^2 + xy + y^2 = 61. \end{cases}$ 16. $\begin{cases} x - y = 4. \\ x^3 - y^3 = 988. \end{cases}$

145. An equation containing two unknown numbers is *symmetrical* when the unknown numbers can change places without changing the equation; as, $3x^2 - 4xy + 3y^2 = 2$; $x^4 + 5x^2y + 5xy^2 + y^4 = -5x^3y^3$.

EXAMPLE 1. Solve the equations: $\begin{cases} x^2 + y^2 = 89 & (1) \\ xy = 40 & (2) \end{cases}$

Process. Add (1) to twice (2), $x^2 + 2xy + y^2 = 169$ (3)
Subtract twice (2) from (1), $x^2 - 2xy + y^2 = 9$ (4)
Extract the square root of (3), $x + y = \pm 13.$
Extract the square root of (4), $x - y = \pm 3.$

We now have to solve the four pairs of simultaneous equations,

$$\left.\begin{matrix} x + y = 13 \\ x - y = 3 \end{matrix}\right\},\quad \left.\begin{matrix} x + y = 13 \\ x - y = -3 \end{matrix}\right\},\quad \left.\begin{matrix} x + y = -13 \\ x - y = 3 \end{matrix}\right\},\quad \left.\begin{matrix} x + y = -13 \\ x - y = -3 \end{matrix}\right\}.$$

There are four pairs of values, two of which are given by $x = \pm 8$, $y = \pm 5$, and the other two by $x = \pm 5$, $y = \pm 8$, in which the upper signs are to be taken together, and the lower signs are to be taken together.

Notes: 1. If the second members of two simple equations have the sign $\pm$, we will have six simultaneous simple equations to consider.

2. The above equations may be solved as in Art. 144, but the symmetrical method is more simple.

EXAMPLE 2. Solve the equations: $\begin{cases} x^3 + y^3 = 126 & (1) \\ x^2 - xy + y^2 = 21 & (2) \end{cases}$

Process. Divide (1) by (2), $x + y = 6$ (3)
Square (3), $x^2 + 2xy + y^2 = 36$ (4)
Subtract (2) from (4), $3xy = 15$, or $xy = 5$ (5)
Subtract (5) from (2), $x^2 - 2xy + y^2 = 16.$
Extract the square root, $x - y = \pm 4$ (6)
Add (3) and (6) and divide the result by 2, $x = 5$, or 1.
Subtract (6) from (3) and divide the result by 2, $y = 1$, or 5.

EXAMPLE 3. Solve the equations: $\begin{cases} x^2 + y^2 - x - y = 78 & (1) \\ xy + x + y = 39 & (2) \end{cases}$

Process. Add (1) to twice (2), $x^2+2xy+y^2+x+y = 156.$

Factor, $(x+y)^2 + (x+y) = 156.$

Regard $x+y$ as the unknown number, complete the square, and solve, $x + y = 12, \text{ or } -13$ (3)

Subtract twice (2) from (1), factor, and transpose $3(x+y)$,

$$(x-y)^2 = 3(x+y) \quad (4)$$

From (3) and (4), $(x-y)^2 = 36, \text{ or } -39.$

Therefore, $x - y = \pm 6, \text{ or } \pm\sqrt{-39}$ (5)

Add (5) and (3), etc., $x = 9, \text{ or } 3, \text{ and } \dfrac{-13 \pm \sqrt{-39}}{2}.$

Subtract (5) from (3), etc., $y = 3, \text{ or } 9, \text{ and } \dfrac{-13 \mp \sqrt{-39}}{2}.$

EXAMPLE 4. Solve the equations: $\begin{cases} x^4 + y^4 = 82 & (1) \\ x + y = 4 & (2) \end{cases}$

Process. Raise (2) to the fourth power,

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 256 \quad (3)$$

Subtract (1) from (3), etc.,

$$2x^3y + 3x^2y^2 + 2xy^3 = 87 \quad (4)$$

Square (2) and multiply the result by $2xy$,

$$2x^3y + 4x^2y^2 + 2xy^3 = 32xy \quad (5)$$

Subtract (4) from (5), etc., $x^2y^2 - 32xy = -87.$

Regard xy as the unknown number, complete the square, and solve, $xy = 29, \text{ or } 3.$

We now have the two pairs of equations to solve,

$$\left.\begin{matrix} x + y = 4 \\ xy = 29 \end{matrix}\right\}, \qquad \left.\begin{matrix} x + y = 4 \\ xy = 3 \end{matrix}\right\}.$$

From the first, $\begin{cases} x = 2 \pm 5\sqrt{-1}, \\ y = 2 \mp 5\sqrt{-1}. \end{cases}$

From the second, $\begin{cases} x = 3, \text{ or } 1. \\ y = 1, \text{ or } 3. \end{cases}$ Hence,

When the Equations are Symmetrical. Combine them in such a manner as to remove the highest powers of x and y.

Exercise 127.

Solve the following equations :

1. $\begin{cases} x^2 + y^2 = 74. \\ x y = 35. \end{cases}$

2. $\begin{cases} x^3 + y^3 = 407. \\ x + y = 11. \end{cases}$

3. $\begin{cases} x^2 + x + y + y^2 = 18. \\ x y = 6. \end{cases}$

4. $\begin{cases} x^2 + y^2 + 2x + 2y = 50. \\ x y + x + y = 23. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 52. \\ x + y + x y = 34. \end{cases}$

6. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{61}{900}. \\ x y = 30. \end{cases}$

7. $\begin{cases} 4x^2 + x y + 4y^2 = 58. \\ 5y^2 + 5x^2 = 65. \end{cases}$

8. $\begin{cases} x^4 + x^2 y^2 + y^4 = 651. \\ x^2 - x y + y^2 = 21. \end{cases}$

9. $\begin{cases} x^4 + x^2 y^2 + y^4 = 931. \\ x^2 + x y + y^2 = 49. \end{cases}$

10. $\begin{cases} x^2 - x y + y^2 = 76. \\ x + y = 14. \end{cases}$

11. $\begin{cases} x^4 + x^2 y^2 + y^4 = 133. \\ x^2 + x y + y^2 = 19. \end{cases}$

12. $\begin{cases} \dfrac{34}{x^2 + y^2} = \dfrac{15}{x y}. \\ x + y = 8. \end{cases}$

146. An algebraic expression is said to be *homogeneous* when all its terms are of the same degree.

Thus, $9x^{10} + 3xy^9 - 8x^6y^4$ is homogeneous, for each term is of the 10th degree and has ten dimensions.

EXAMPLE 1. Solve the equations : $\begin{cases} 6x^2 + 2y^2 - 5xy = 12 & (1) \\ 3x^2 + 2xy = 3y^2 - 3 & (2) \end{cases}$

Process. Let $y = vx$, and substitute in both equations.

From (1), $\quad 6x^2 + 2v^2x^2 - 5vx^2 = 12.$

Therefore, $$x^2 = \frac{12}{6 - 5v + 2v^2} \qquad (3)$$

From (2), $\quad 3x^2 + 2vx^2 = 3v^2x^2 - 3.$

Therefore, $$x^2 = \frac{3}{3v^2 - 2v - 3} \qquad (4)$$

Equate (3) and (4), $$\frac{12}{6 - 5v + 2v^2} = \frac{3}{3v^2 - 2v - 3}$$

Simplify and solve for v, $\quad v = \frac{3}{2}$, or $-\frac{6}{5}$.

Substitute $v = \frac{3}{2}$ in (3),

$$x^2 = \frac{12}{6 - 5 \times \frac{3}{2} + 2(\frac{3}{2})^2} = 4.$$

$\therefore\ x = \pm 2.$

$y = \frac{3}{2} x = \pm 3.$

Substitute $v = -\frac{6}{5}$ in (3),

$$x^2 = \frac{12}{6 - 5 \times -\frac{6}{5} + 2(-\frac{6}{5})^2} = \frac{25}{31}.$$

$\therefore\ x = \pm \frac{5}{31}\sqrt{31}.$

$y = -\frac{6}{5} x = \mp \frac{6}{31}\sqrt{31}.$

Notes: 1. In finding the last values of x and y, it will be observed that $\pm$ values of x gives respectively $-$ and $+$ values of y. This indicates that the equations can be satisfied only by making $y = -\frac{6}{31}\sqrt{31}$, when $x = +\frac{5}{31}\sqrt{31}$; and when $x = -\frac{5}{31}\sqrt{31}$, y must be $+\frac{6}{31}\sqrt{31}$.

2. The sign $\mp$ denotes precedence of the negative value.

When each Equation is of the Second Degree and Homogeneous. Substitute vx for y in both equations.

Exercise 128.

Solve the following equations:

1. $\begin{cases} x^2 + xy = 15. \\ y^2 + xy = 10. \end{cases}$

2. $\begin{cases} x^2 - xy = 24. \\ xy - y^2 = 8. \end{cases}$

3. $\begin{cases} x^2 + 4xy = 133. \\ 4xy + 16y^2 = 228. \end{cases}$

4. $\begin{cases} 2x^2 + 3xy = 26. \\ 3y^2 + 2xy = 39. \end{cases}$

5. $\begin{cases} 4x^2 + 4xy + 4y^2 = 13. \\ 8x^2 - 12xy + 8y^2 = 11. \end{cases}$

6. $\begin{cases} x^2 - 3xy + y^2 = -1. \\ 3x^2 - xy + 3y^2 = 13. \end{cases}$

7. $\begin{cases} 2x^2 - 5xy + 3y^2 = 1. \\ 3x^2 - 5xy + 2y^2 = 4. \end{cases}$

8. $\begin{cases} x^2 - 2xy = 21. \\ xy + y^2 = 18. \end{cases}$

9. $\begin{cases} x^2 + 3xy = 54. \\ xy + 4y^2 = 115. \end{cases}$

10. $\begin{cases} x^2 + xy + 2y^2 = 74. \\ 2x^2 + 2xy + y^2 = 73. \end{cases}$

Queries. What is a homogeneous equation? Into what forms may simultaneous quadratic equations, which can be solved, be grouped? What is the degree of the equation arising from eliminating one unknown number from two equations, each of the second degree? Prove it. How may such equations be solved?

Note. In solving the following equations the student is cautioned not to work at random, but to study the equations until he sees how they may be combined in order to produce simple equations, and then perform the operations thus suggested. Usually the operations of addition, subtraction, multiplication, division, or factoring will effect a simplification of the equations.

Miscellaneous Exercise 129.

Solve the following equations:

1. $\begin{cases} x + y = 8. \\ xy = 15. \end{cases}$

2. $\begin{cases} x - y = 3. \\ xy = 18. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 113. \\ x - y = 1. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 244. \\ x + y = 22. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 74. \\ xy = 35. \end{cases}$

6. $\begin{cases} x^2 y - 7x = -2. \\ (y-1)x^2 - 3x = 2. \end{cases}$

7. $\begin{cases} x^2 - y^2 = 175. \\ x - y = 5. \end{cases}$

8. $\begin{cases} x^2 + 5y^2 = 6x. \\ x^2 - 5y^2 = 4xy. \end{cases}$

9. $\begin{cases} x^2 + xy = 140. \\ y^2 + xy = 56. \end{cases}$

10. $\begin{cases} x^2 - xy = 9. \\ xy - y^2 = 4. \end{cases}$

11. $\begin{cases} x^3 + 3x^2y + 3xy^2 + 2y^3 = 0. \\ x^2 + xy + y^2 = 1 - x^2y^2. \end{cases}$

12. $\begin{cases} .1y + .125x = y - x. \\ y - .5x = .75xy - 3x. \end{cases}$

13. $\begin{cases} .3x + .125y = 3x - y. \\ 3x + y = -2.25xy. \end{cases}$

14. $\begin{cases} x^4 + y^4 = 706. \\ x + y = 2. \end{cases}$

15. $\begin{cases} x + y + x^2 + y^2 = 18. \\ xy = 6. \end{cases}$

16. $\begin{cases} 4(x+y) = 3xy. \\ x + y + x^2 + y^2 = 26. \end{cases}$

17. $\begin{cases} x^4 + y^4 = 337. \\ x + y = 7. \end{cases}$

18. $\begin{cases} x^2 + xy + x = 14. \\ y^2 + xy + y = 28. \end{cases}$

19. $\begin{cases} x^3 - y^3 = 208. \\ xy(x-y) = 48. \end{cases}$

20. $\begin{cases} xy + x^2y = 12. \\ y + x^3y = 18. \end{cases}$

21. $\begin{cases} x^2 + y^2 = 8xy. \\ x + y = 5. \end{cases}$

22. $\begin{cases} 2xy + 12 = 3x^2. \\ 6xy + 12 = x^4. \end{cases}$

23. $\begin{cases} \dfrac{2}{x} + \dfrac{3}{y} = 4. \\ xy = 2. \end{cases}$

24. $\begin{cases} \dfrac{x + y}{1 + xy} = a. \\ \dfrac{x - y}{1 - xy} = b. \end{cases}$

25. $\begin{cases} x^4 + x^2y^2 + y^4 = 7371. \\ x^2 - xy + y^2 = 63. \end{cases}$

26. $\begin{cases} x^4 + y^4 = 641. \\ xy(x^2 + y^2) = 290. \end{cases}$

27. $\begin{cases} x^2 + 3xy + y^2 = 19. \\ x^2 + y^2 = 10. \end{cases}$

28. $\begin{cases} x^2 - 3xy + y^2 = -5. \\ 3x^2 - 5xy + 9y^2 = 9. \end{cases}$

29. $\begin{cases} y^3 - x^3 = a^3. \\ y - x = a. \end{cases}$

30. $\begin{cases} x^4 + y^4 = 14x^2y^2. \\ x + y = a. \end{cases}$

31. $\begin{cases} 96 - x^2y^2 = 4xy. \\ x + y = 6. \end{cases}$

32. $\begin{cases} x^3 - y^3 = 56. \\ x^2 + xy + y^2 = 28. \end{cases}$

33. $\begin{cases} y^2 - xy = a^2 + b^2. \\ xy - x^2 = 2ab. \end{cases}$

34. $\begin{cases} x^3y - y = 21. \\ x^2y - xy = 6. \end{cases}$

35. $\begin{cases} \dfrac{x^4}{y^2} + \dfrac{2x^2}{y} = \dfrac{480}{49}. \\ x^2 + y^2 = 65. \end{cases}$

36. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{x + y}{6}. \\ \dfrac{x + y}{6} = \dfrac{5}{x + y + 1}. \end{cases}$

37. $\begin{cases} x^2y^2 + 5xy = 84. \\ x + y = 8. \end{cases}$

38. $\begin{cases} x + y + \sqrt{x + y} = 12. \\ x^2 + y^2 = 41. \end{cases}$

39. $\begin{cases} x + y + \sqrt{x + y} = 12. \\ x^3 + y^3 = 189. \end{cases}$

40. $\begin{cases} x + \sqrt{xy} + y = 19. \\ x^2 + xy + y^2 = 133. \end{cases}$

41. $\begin{cases} \sqrt{x} + \sqrt{y} = 4. \\ \sqrt{x^3} + \sqrt{y^3} = 28. \end{cases}$

42. $\begin{cases} x^{\frac{1}{4}} + y^{\frac{1}{4}} = 6. \\ x^{\frac{3}{4}} + y^{\frac{3}{4}} = 126. \end{cases}$

43. $\begin{cases} y^2 - x^2 = 4ab. \\ xy = a^2 - b^2. \end{cases}$

44. $\begin{cases} 2x^2 - 3xy + y^2 = 4. \\ 2xy - 3y^2 - x^2 = -9. \end{cases}$

45. $\begin{cases} \dfrac{1}{x^3} + \dfrac{1}{y^3} = \dfrac{126}{125}. \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{6}{5}. \end{cases}$

48. $\begin{cases} \dfrac{1}{x^3} - \dfrac{1}{y^3} = 91. \\ \dfrac{1}{x} - \dfrac{1}{y} = 1. \end{cases}$

46. $\begin{cases} \dfrac{y}{x} + 4\sqrt{\dfrac{y}{x}} = \dfrac{33}{4}. \\ y - x = 5. \end{cases}$

49. $\begin{cases} \dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{10}{3}. \\ x^2 + y^2 = 45. \end{cases}$

47. $\begin{cases} \dfrac{1}{x^2} - \dfrac{1}{y^2} = 3. \\ \dfrac{1}{x} + \dfrac{1}{y} = 21. \end{cases}$

50. $\begin{cases} \dfrac{x+y}{a+b} = \dfrac{x-y}{a-b}. \\ xy = ab. \end{cases}$

51. The sum of the squares of the digits composing a number of two places of figures is 25, and the product of the digits is 12. Find the number.

52. There are two numbers whose sum, multiplied by the greater, gives 144, and whose difference, multiplied by the less, gives 14. Find the numbers.

53. The sum of the squares of two numbers is a, and the difference of their squares is b. Find the numbers. Solve for $a = 170$ and $b = 72$.

54. A number divided by the product of its two digits gives the quotient 2; and if 27 be added to the number, the digits are reversed. Find the number.

55. The sum of two numbers is a, and the sum of their 4th powers is b. Find the number. Solve for $a = 4$ and $b = 82$.

56. The difference of two numbers is a, and the difference of their cubes is $7\,a^3$ Find the numbers.

57. The difference of two numbers is 3, and the difference of their 5th powers is 3093. Find the numbers.

58. A number consisting of two digits has one decimal place; the difference of the squares of the digits is 20, and if the digits be reversed, the sum of the two numbers is 11. Find the number.

59. Find three numbers whose sum is 38, such that the difference of the first and second shall exceed the difference of the second and third by 7, and the sum of whose squares is 634.

60. The small wheel of a bicycle makes 135 revolutions more than the larger wheel in a distance of 260 yards; if the circumference of each were one foot more, the small wheel would make 27 revolutions more than the large wheel in a distance of 70 yards. Find the number of feet in the circumference of each wheel.

61. Find two numbers such that their difference shall be a, and the product of their nth roots c. Solve for $a = 4$, $c = 2$, and $n = 5$.

62. Find a fraction such if the numerator be increased and the denominator diminished by 2, the result will be its reciprocal; while if the numerator be diminished and the denominator increased by 2, the result will be $\frac{16}{15}$ less than its reciprocal.

63. A principal of $10,000 amounts, with simple interest, to $14,200 after a certain number of years. Had the rate of interest been 1 % higher and the time 1 year longer, it would have amounted to $15,600. Find the time and rate.

64. A sum of money at interest amounted at the end of the year to $10,920. If the rate of interest had been 1 % less, and the principal $100 more, the amount would have been the same. Find the principal and rate of interest.

CHAPTER XXIV.

INDETERMINATE EQUATIONS.

147. SIMPLE **Indeterminate Equations** are equations of the first degree that admit of an unlimited number of solutions.

Thus, in $3x-2y=2$, if $y=2$, $x=2$; if $y=3$, $x=2\frac{2}{3}$; if $y=5$, $x=4$; if $y=8$, $x=6$; etc. It is evident that an unlimited number of values may be given to y and x that will satisfy the equation. Hence, an equation containing two unknown numbers admits of as many solutions as we please, and is indeterminate.

Since the values of the unknown numbers are dependent upon each other, they may be confined to a particular limit; as, for example, suppose the variables to be restricted to *positive* or *negative* integers, we may thus limit the *number* of solutions.

EXAMPLE 1. Solve $19x+5y=119$, in positive integers.

Solution. Transpose $19x$, $5y=119-19x$.

Therefore, $$y=23-3x+4\left(\frac{1-x}{5}\right) \quad (1)$$

Since the value of y is to be integral, then $\frac{1-x}{5}$ *must be integral,* although fractional in *form;* and so also is any multiple of it.

Let $\frac{1-x}{5}=n$, an integer.

Therefore, $$x=1-5n \quad (2)$$

Substitute in (1), $$y=20+19n \quad (3)$$

We must take only such integral values for n as will give *positive integral* values for x and y.

(2) shows that n may be 0, or have any negative integral value, but cannot have a positive integral value.

(3) shows that n may be 0 and -1, but cannot have a negative integral value greater than 1.

Therefore, n may be 0 and -1.

Hence, $\left.\begin{array}{l} x = 1 \\ y = 20 \end{array}\right\}$, and $\left.\begin{array}{l} x = 6 \\ y = 1 \end{array}\right\}$.

Query. Can n be -2 or $+1$? Why?

EXAMPLE 2. Solve $7x - 12y = 19$, in positive integers.

Process. Transpose and solve for x, $x = 2 + y + 5\left(\frac{1+y}{7}\right)$ (1)

Let $\frac{1+y}{7} = n$, an integer.

Therefore, $y = 7n - 1$ (2)

Substitute in (1), $x = 12n + 1$ (3)

Evidently x and y will both be positive integers if n have *any* positive integral value.

Hence, $x = 13, 25, 37, 49, \ldots.$
$y = 6, 13, 20, 27, \ldots.$

Notes: 1. Having obtained a few of the possible values of x and y, the law will become evident.

2. It will be seen from the above solutions that when only positive integral values are required, the number of solutions will be limited or unlimited according as the sign connecting the terms is positive or negative.

EXAMPLE 3. Solve $190x - 23y = 708$, in least positive integers.

Process. Solve for y, $y = 8x - 30 - 6\left(\frac{3-x}{23}\right)$ (1)

Let $\frac{3-x}{23} = n$, an integer.

Therefore, $x = 3 - 23n$ (2)

Substitute in (1), $y = -6 - 190n$ (3)

Evidently x and y will both be least positive integers if n be -1.

Therefore, $n = -1$, $x = 26$, and $y = 184$.

Note 3. If the coefficient of the unknown number in the numerator of the fraction is not 1, it will be necessary to make several transformations.

EXAMPLE 4. Solve $21x + 17y = 2000$, in positive integers.

Solution. Solve for y, $$y = 117 - x + \frac{11 - 4x}{17} \quad (1)$$

Transpose, $$y + x - 117 = \frac{11 - 4x}{17}.$$

Since x and y are to be integral, $y + x - 117$ will be integral; hence, $\frac{11 - 4x}{17}$ will be integral.

Let $$\frac{11 - 4x}{17} = n, \text{ an integer.}$$

Therefore, $$x = 2 - 4n + \frac{3 - n}{4} \quad (2)$$

Now $\frac{3 - n}{4}$ *must* be integral.

Let $$\frac{3 - n}{4} = m, \text{ an integer.}$$

Therefore, $$n = 3 - 4m.$$

Substitute in (2), $$x = 17m - 10 \quad (3)$$

Substitute in (1), $$y = 130 - 21m \quad (4)$$

(3) shows that m may have any positive integral value, but cannot be 0, or have any negative integral value.

(4) shows that m may have any integral value from 0 to 6, or any negative integral value, but cannot have a positive integral value greater than 6.

Therefore, m may be 1, 2, 3, 4, 5, 6, giving the following pairs of values:

$$x = \quad 7,\ 24,\ 41,\ 58,\ 75,\ 92.$$
$$y = 109,\ 88,\ 67,\ 46,\ 25,\ \ 4.$$ Hence,

To Solve a Simple Indeterminate Equation, Involving Two Unknown Numbers, for Integral Values. Find the value of one of the unknown numbers. Place the fractional part of this value equal to n, an integer, and solve the resulting equation for the other unknown number. Substitute this result in the value first obtained. Solve the two simple equations thus formed, by inspection, for integral values of n.

Notes: 4. It is better, in solving the original equation, to solve for the unknown number which has the least coefficient.

5. A little ingenuity in arranging the terms will often obviate the necessity of a second transformation.

148. There can be no integral values of x and y in an equation of the form $ax \pm by = c$, if a and b have a common factor not common also to c.

For, suppose d to be any factor of a and also of b, but not of c, such that $a = md$ and $b = nd$.

Then $mdx \pm ndy = c$, or $mx \pm ny = \frac{c}{d}$.

Since m and n are integers, if x and y be also integers, $mx \pm ny$ is an integer. But $\frac{c}{d}$ is a fraction. Hence, no integral values of x and y can be found.

Notes: 1. If a, b, and c have a common factor, it should be removed by division, then proceed as in Art. 147.

2. The solution of any indeterminate equation of the form $ax - by = \pm c$, in which a and b are prime to each other, is always possible, and admits of an unlimited number of integral solutions (Ex. 2, Art. 147). If the equation be of the form $ax + by = c$, the number of results will always be limited; and, in some cases, the solution is impossible (Ex. 1, Art. 147).

Exercise 130.

Solve in positive integers:

1. $2x + 3y = 25$; $14x = 5y - 7$; $3x = 8y - 16$.
2. $5x + 11y = 254$; $9x + 13y = 2000$.
3. $15x - 17y = 1$; $13x - 9y = 1$; $9x - 13y = 10$.

Solve in least positive integers:

4. $3x + 7y = 39$; $3x + 4y = 39$; $7x + 15y = 225$.
5. $27x - 19y = 43$; $2x + 7y = 125$; $555y - 22x = 73$.
6. $19x - 5y = 119$; $17x = 49y - 8$.

Are integral solutions possible for the following? Why?

7. $3x + 21y = 1000$; $7x + 14y = 71$.
8. $323x - 527y = 1000$; $166x - 192y = 91$.

9. Solve $7x + 15y = 145$, in positive integers, so that x may be a multiple of y.

Suggestion. Let $x = ny$, then $y = \frac{145}{7n+15}$, and $x = \frac{145n}{7n+15}$.

10. Solve $39x - 6y = 12$, in positive integers, so that y may be a multiple of x.

11. Solve $20x - 31y = 7$, so that x and y may be positive, and their sum an integer.

Suggestion. Put $x + y = n$.

149. A problem is indeterminate when it involves less conditions than there are unknown numbers.

Exercise 131.

1. Find a number which being divided by 3, 4, and 5, gives the remainders 2, 3, and 4, respectively.

Solution. Let x represent the number and y the sum of the quotients, then,

$$\frac{x-2}{3} + \frac{x-3}{4} + \frac{x-4}{5} = y.$$

Simplify and solve for x, $\quad x = y + 2 + 13\left(\frac{3+y}{47}\right) \quad (1)$

Let $\quad \frac{3+y}{47} = n$, an integer.

Therefore, $\quad y = 47n - 3.$

Substitute in (1), $\quad x = 60n - 1.$

Hence, n may be 1, 2, 3, 4, etc.

Therefore, $\quad x = 59, 119, 179, 239$, etc.

$\quad y = 44, 91, 138, 185$, etc.

2. Find the least number which being divided by 2, 3, 4, 5, and 6, gives remainders 1, 2, 3, 4, and 5, respectively.

3. Find two numbers which, multiplied respectively by 14 and 18, have for the sum of their products 200.

4. Divide 142 into two parts, one of which is divisible by 9, and the other by 14.

5. There are two unequal rods, one 5 feet long and the other 7. How many of each can be taken to make up a length of 123 feet?

6. Find two fractions having 5 and 7 for denominators, and whose sum is $\frac{26}{35}$.

7. Find the least number that when divided by 9 and 17 will give remainders 5 and 12, respectively.

Suggestion. Let N represent the number, $\dfrac{N-5}{9}=x$, and $\dfrac{N-12}{17}=y$. $\therefore\ 9x=17y+7$.

8. A farmer bought sheep, pigs, and hens. The whole number bought is 125, and the whole price, \$225. The sheep cost \$5, the pigs \$2.50, and the hens 25 cents. How many of each did he buy?

Solution. Let $x =$ the *number* of sheep,
$y =$ the *number* of pigs,
and $z =$ the *number* of hens.

Then, $$x+y+z=125 \quad (1)$$
and $$5x+2.5y+.25z=225 \quad (2)$$

From (1) and (2), $$y=86-2x-\frac{x-1}{9} \quad (3)$$

Let $$\frac{x-1}{9}=n, \text{ an integer.}$$

Therefore, $$x=9n+1.$$

Substitute in (3), $$y=84-19n.$$

Substitute in (1), $$z=40+10n.$$

Therefore, n may be 0, 1, 2, 3, and 4, giving the following values:

$$x=\ \ 1,\ 10,\ 19,\ 28,\ 37.$$
$$y=84,\ 65,\ 46,\ 27,\ \ 8.$$
$$z=40,\ 50,\ 60,\ 70,\ 80.$$

Queries. How many solutions? In how many different ways may the stock be bought? How solve by means of only two unknown numbers?

9. How can one pay a sum of \$1.50 with 3 and 5 cent pieces?

10. Can a grocer put up the worth of \$3.50 in 11 and 7 cent sugar? In how many ways can he do it in even and odd pounds, respectively? Find the greatest and least number of pounds of the 7-cent sugar he can use.

11. Is it possible to pay £50 by means of guineas and three-shilling pieces only?

12. A owes B \$5.15. A has only 50-cent pieces and B only 3-cent pieces. How may they settle the account?

13. A farmer bought horses at \$60 a head and sheep at \$8, and found that he had invested \$4 more in sheep than horses. How many of each kind did he buy?

14. A farmer invested \$1000 in 75 head of cattle, worth \$25, \$15, and \$10 per head. Find the number of each kind, and the number of ways in which he could buy them.

15. A grocer had an order for 75 pounds of tea at 55 cents a pound, but having none at that price he mixed some at 30 cents, some at 45 cents, and some at 80 cents. How much of each kind did he use, and in how many ways can he mix it?

16. How many pounds of 20, 35, and 40 cent coffee must a grocer take to make a mixture of 150 pounds worth 30 cents a pound? In how many ways can the mixture be made?

17. How many gallons of \$1.50, \$1.90, and \$1.20 wine must a vintner take to make a mixture of 40 gallons worth \$1.60 per gallon? How many ways may the mixture be made? Can an odd number of gallons of each kind be taken? An even number?

18. In how many ways can £1 be paid in half-crowns, shillings, and sixpence, the number of coins in each payment being 18?

19. A hardware merchant paid \$180 for 20 stoves. There were three sizes: one \$19 each, another \$7, the other \$6. How many of each size did he buy?

20. A person having a basket of oranges, containing between 50 and 72, takes them out 4 at a time, and finds 1 over; he then takes them out 3 at a time, and finds none over. How many had he?

Suggestion. Let N represent the number, $\frac{N-1}{4}=x$; and $\frac{N}{3}=y$.
$\therefore\ y=x+\frac{1+x}{3}$. Put $\frac{1+x}{3}=n$. Then n must be 5 or 6.

21. A poultry dealer has a basket containing between 200 and 300 eggs, he finds that when he sells them 13 at a time there are 9 over, but when he sells them 17 at a time there are 14 over. Find the number of eggs.

22. Two countrymen together have 100 eggs. If the first counts his by eights and the second his by tens, there is a surplus of 7 in each case. How many eggs has each?

23. A surveyor has three ranging poles of lengths 7 feet, 10 feet, and 12 feet. How may he take 40 of them to measure 113 yards? In how many ways may the measurement be made?

CHAPTER XXV.

INEQUALITIES.

150. Since a positive number is greater than any negative number, the statement that a is algebraically greater than b, or that $a - b$ is positive, is expressed by $a > b$; that a is algebraically less than b, or that $a - b$ is negative, is expressed by $a < b$. Hence,

An **Inequality** is a statement that one expression is greater or less than another; as,

$$1 - x^2 > \frac{1 - 2x}{x^2}; \quad m - n < x.$$

The expression at the left of the sign is called the **first member,** and the expression at the right, the **second member** of the inequality.

The form $a > b > c$, means that b is less than a but greater than c.

Notes: 1. Inequalities are said to subsist in the **same sense** when the first member is the greater in each, or the first member is the less in each; as, $3 > 2$, $7 > 5$, and $5 > 3$; $a < b$, $c < d$, and $m < n$.

2. Two inequalities are said to subsist in a **contrary sense** when the first member is the greater in one, and the less in the other; as, $5 > 3$ and $a < b$; $m < 5$ and $b > n$.

3. An inequality is said to be solved when the limit to the value of the unknown number is found.

151. Subtract $a + b$ from each member of $a > b$,

then, $$a - (a + b) > b - (a + b).$$

Simplify, $$-b > -a,$$

or, $$-a < -b. \quad \text{Hence,}$$

I. *If each member of an inequality has its sign changed, the sign of inequality will be reversed.*

Multiply each member of $-5 < 5$ by -2,
then, $10 > -10$.
Multiply each member of $a > b$ by $-m$,
then, $-am < -bm$.
Divide each member of $-6 < 4$ by -2,
then, $3 > -2$.
Divide each member of $a > b$ by $-m$,
then, $-\frac{a}{m} < -\frac{b}{m}$. Hence,

II. *If each member of an inequality be multiplied or divided by the same negative number, the inequality will be reversed.*

Suppose $a > b$, $c > d$, $m > n$, ...
By definition, $a - b$, $c - d$, $m - n$, ... are positive.
Add, $(a - b) + (c - d) + (m - n) + \dots$ is positive.
or, $(a + c + m + \dots) - (b + d + n + \dots)$ is positive.
Therefore (by definition), $a + c + m + \cdots > b + d + n + \dots$.

$$\begin{array}{ll} \text{Thus,} & 7 > 3 \\ & 5 > 2 \\ & \underline{4 > 1} \\ \text{Add,} & 16 > 6, \text{ or divide by 2, } 8 > 3. \quad \text{Hence,} \end{array}$$

III. *If the corresponding members of several inequalities be added, the sum of the greater members will exceed the sum of the lesser members.*

Suppose $a > b$ and $m > n$, then $a - b$ and $m - n$ are positive.
But, $(a - b) - (m - n)$, or $(a - m) - (b - n)$ may be either positive, negative, or 0.
Therefore, $a - m > b - n$, $a - m < b - n$, or $a - m = b - n$.

$$\begin{array}{ll|ll|ll} \text{Thus,} & 5 > 3 & & 7 > 4 & & 8 > 7 \\ & \underline{3 > 2} & & \underline{5 > 1} & & \underline{6 > 5} \\ \text{Subtract,} & 2 > 1 & \text{Subtract,} & 2 < 3 & \text{Subtract,} & 2 = 2. \quad \text{Hence,} \end{array}$$

IV. *If the members of one inequality be subtracted from the corresponding members of another, the resulting inequality will not always subsist in the same sense.*

EXAMPLE 1. Solve $3\frac{1}{3}x - \frac{1-2x}{5} > \frac{3x}{2} + \frac{64}{15}$ for the limits of x.

Solution. Free from fractions and simplify,

$$112x - 6 > 45x + 128.$$

Subtract $45x - 6$, $\quad 67x > 134.$

Divide by 67, $\quad x > 2.$

Therefore, x is greater than 2.

EXAMPLE. 2. Solve the following:

$$\begin{cases} 5x - 3y > 3x + 5 & (1) \\ 3x + y = 22 & (2) \end{cases}$$

Solution. Subtract $3x$ from (1), $\quad 2x - 3y > 5 \quad (3)$

Multiply (2) by 3, $\quad 9x + 3y = 66 \quad (4)$

Add (3) and (4), $\quad 11x > 71.$

Divide by 11, $\quad x > 6\frac{5}{11}.$

From (4), $\quad x = \frac{22 - y}{3}.$

Substitute in (3) and simplify, $\quad -y > -\frac{29}{11}.$

Therefore, $\quad y < 2\frac{7}{11}$ (see I)

EXAMPLE 3. Solve the inequalities:

$$\begin{cases} nx - mn > n^2 - mx & (1) \\ mx - nx + mn < m^2 & (2) \end{cases}$$

Process. Simplify (1) and solve, $\quad x > n.$

Simplify (2) and solve, $\quad x < m.$

Hence, x is greater than n and less than m.

Note. The principles applied to the solutions of equations may be applied to inequalities, except that **if each member of an inequality has its sign changed, the sign of inequality will be reversed.**

Exercise 132.

Find the limit of x in the following:

1. $4x - 3 > \frac{5}{3}x - \frac{5}{6}$; $\frac{8}{7} - \frac{6}{5}x < 9 - 3x$.

2. $\frac{3}{2}x + \frac{2}{3} > \frac{2}{3}x + \frac{1}{2}$; $\frac{x^2 - a}{bx} - \frac{a - x}{b} > \frac{2x}{b} - \frac{a}{x}$.

3. $\dfrac{3ax-2b}{3b}-\dfrac{ax-a}{2b}<\dfrac{ax}{b}-\dfrac{2}{3}.$

4. If $x^2+4x>12$, show that $x>2$.

5. If $7x^2-3x<160$, show that $x<5$.

6. If $4x+12-x^2>0$, show that x is included between 6 and -2.

7. If $9x<20x^2+1$, show that $x>\frac{1}{4}$ or $<\frac{1}{5}$.

8. If $15-x-2x^2>0$, show that x lies between $\frac{5}{2}$ and -3.

Find an integral value of x in the following:

9. $\begin{cases} 2x>30-4x. \\ 10x<3x+49. \end{cases}$

10. $\begin{cases} 1\frac{1}{3}x<\frac{2}{3}x+3\frac{1}{3}. \\ 6x>24-2x. \end{cases}$

11. $\begin{cases} \frac{3}{4}(x+2)+x<\frac{3}{2}(x-4)+9. \\ \frac{1}{4}(x+2)+\frac{1}{3}x>\frac{1}{2}(x+1)+\frac{1}{3}. \end{cases}$

Find the limits of x and y in the following:

12. $\begin{cases} 3x+5y>121. \\ 4x+7y=168. \end{cases}$

13. $\begin{cases} 7x+5y>19. \\ x-y=1. \end{cases}$

14. $\begin{cases} (a+b)x-(a-b)y>4ab. \\ (a-b)x+(a+b)y=2(a+b)(a-b). \end{cases}$

15. A certain number plus 5, is greater than one third the number plus 55; while its half plus 2, is less than 41. Find the number.

16. Find the price of oranges per dozen, when three times the price of one orange, decreased by three cents, is more than twice its price increased by one cent; and eight times the price of one orange, decreased by twenty cents, is less than three times its price increased by ten cents.

152. Since the square of a negative number is positive, if a and b represent any two numbers, $(a-b)^2$ must be positive, *whatever the values of* a *and* b. Therefore, since every positive number is greater than zero,

$$(a-b)^2 > 0.$$

Expand, $$a^2 - 2ab + b^2 > 0.$$

Add $2ab$ to each member, $$a^2 + b^2 > 2ab.$$ Hence,

The sum of the squares of two unequal numbers is greater than twice their product.

Note. The above is a fundamental principle in inequalities.

EXAMPLE 1. Show that $a^2 + b^2 + c^2 > ab + ac + bc$, a and b positive.

Proof. Since a, b, and c are any unequal numbers,

$$a^2 + b^2 > 2ab \quad (1)$$

$$a^2 + c^2 > 2ac \quad (2)$$

$$b^2 + c^2 > 2bc \quad (3)$$

Add the corresponding members of (1), (2), and (3),

$$2a^2 + 2b^2 + 2c^2 > 2ab + 2ac + 2bc.$$

Divide by 2, $$a^2 + b^2 + c^2 > ab + ac + bc.$$

Query. How if $a = b = c$?

EXAMPLE 2. Show that $a^3 + b^3 > a^2b + ab^2$.

Proof. We shall have, $$a^3 + b^3 > a^2b + ab^2.$$

Factor, $$(a+b)(a^2 - ab + b^2) > ab(a+b).$$

Divide by $a+b$, $$a^2 - ab + b^2 > ab.$$

Add ab, $$a^2 + b^2 > 2ab.$$

Therefore, $$a^3 + b^3 > a^2b + ab^2.$$

EXAMPLE 3. Which is the greater, $\sqrt{\frac{a^2 b^2}{mn}} + \sqrt{\frac{m^2 n^2}{ab}}$ or $\sqrt{ab} + \sqrt{mn}$?

Proof. We shall have,

$$\sqrt{\frac{a^2 b^2}{mn}} + \sqrt{\frac{m^2 n^2}{ab}} > \text{ or } < \sqrt{ab} + \sqrt{mn}.$$

Square each member,

$$\frac{a^2 b^2}{mn} + 2\sqrt{abmn} + \frac{m^2 n^2}{ab} > \text{ or } < ab + 2\sqrt{abmn} + mn.$$

Subtract $2\sqrt{abmn}$, $\quad \frac{a^2 b^2}{mn} + \frac{m^2 n^2}{ab} > \text{ or } < ab + mn.$

Free from fractions and factor,

$$(ab + mn)(a^2 b^2 - abmn + m^2 n^2) > \text{ or } < abmn(ab + mn).$$

Divide by $ab + mn$,

$$a^2 b^2 - abmn + m^2 n^2 > \text{ or } < abmn.$$

Add $abmn$, $\quad a^2 b^2 + m^2 n^2 > \text{ or } < 2abmn.$

But, $\quad a^2 b^2 + m^2 n^2 > 2abmn.$

Therefore, $\quad \sqrt{\frac{a^2 b^2}{mn}} + \sqrt{\frac{m^2 n^2}{ab}} > \sqrt{ab} + \sqrt{mn}.$

Exercise 133.

Show that, the letters being unequal, positive, and integral:

1. $a^2 + 3b^2 > 2ab + 2b^2$; $\frac{a}{b^2} + \frac{b}{a^2} > \frac{1}{a} + \frac{1}{b}$.

2. $ab^{-2} + a^{-2}b > a^{-1} + b^{-1}$; $(m^2 + n^2)(m^4 + n^4) > (m^3 + n^3)^2$.

3. $xy + xz + yz < (x + y - z)^2 + (x + z - y)^2 + (y + z - x)^2$.

Which is the greater:

4. $n^3 + 1$ or $n^2 + n$; $\frac{a+b}{2}$ or $\frac{2ab}{a+b}$; $\frac{m}{n^2} - \frac{n}{m^2}$ or $\frac{1}{m} - \frac{1}{n}$.

5. $\frac{x+y}{x-y}$ or $\frac{x^2+y^2}{x^2-y^2}$; $3(1 + a^2 + a^4)$ or $(1 + a + a^2)^2$.

6. a^3+2b^3 or $3ab^2$; $\dfrac{\sqrt{2}}{\sqrt[3]{3}}$ or $\dfrac{\sqrt{3}}{\sqrt[3]{5}}$; $\sqrt{2}+\sqrt{7}$ or $\sqrt{3}+\sqrt{5}$

Queries. How in 4 and 6, if $a=b$? In 4, if $n=1$?

7. If $a^2+b^2+c^2=1$, and $x^2+y^2+z^2=1$, show that $ax+by+cz<1$.

Query. How if $a=b=c=x=y=z$?

If $a>b$, show that:

8. $a-b>(\sqrt{a}-\sqrt{b})^2$; $a^3+7a^2b>(a+b)^3$.

9. $a^a b^b > a^b b^a$; $a^3+13ab^2>5a^2b+9b^3$.

Miscellaneous Exercise 134.

EXAMPLE 1. Solve the inequalities:

$$\begin{cases}\sqrt{2(xy+y^2)+4}<\sqrt{(2y-1)(y+x)} & (1)\\ 2x+5y>8 & (2)\end{cases}$$

Solution. Square each member of (1) and simplify,

$$2xy+2y^2+4<2y^2+2xy-y-x.$$

Subtract $2xy+2y^2$,
$$4<-y-x \qquad (3)$$
Multiply each member of (3) by 2,
$$8<-2y-2x \qquad (4)$$
Add the corresponding members of (2) and (4),
$$3y>16. \quad \therefore\ y>5\tfrac{1}{3}.$$
Multiply each member of (3) by 5,
$$20<-5y-5x \qquad (5)$$
Add (2) and (5),
$$-3x>28, \text{ or } 3x<-28. \quad \therefore\ x<-9\tfrac{1}{3}.$$

EXAMPLE 2. Simplify $(y+x<m-n)(m^2+mn+n^2>y-x)$.

Solution. We are to multiply the corresponding members together,
$$(y+x)(y-x)=y^2-x^2.$$
$$(m-n)(m^2+mn+n^2)=m^3-n^3.$$
Therefore, $(y+x<m-n)(m^2+mn+n^2>y-x)=y^2-x^2<m^3-n^3.$

EXAMPLE 3. Which is the greater, $x^5 + y^5$ or $x^4 y + y^4 x$?

Proof. We shall have, $x^5 + y^5 >$ or $< x^4 y + y^4 x$.

Subtract $x^4 y + y^4 x$, $x^5 - x^4 y + y^5 - y^4 x >$ or < 0.

Factor, $(x^4 - y^4)(x - y) >$ or < 0.

Now, whether $x >$ or $< y$, the two factors, $x^4 - y^4$ and $x - y$, will have the *same sign*. Hence, since $(x^4 - y^4)(x - y)$ is always positive,

$$(x^4 - y^4)(x - y) > 0.$$

Therefore, $x^5 + y^5 > x^4 y + y^4 x$.

EXAMPLE 4. Which is the greater, $m^4 - n^4$ or $4n^3(m - n)$ when $m > n$?

Proof. We shall have, $m^4 - n^4 >$ or $< 4m^3(m - n)$.

Divide by $m - n$, $m^3 + m^2 n + m n^2 + n^3 >$ or $< 4m^3$.

Subtract $m^3 + m^2 n$ and factor the resulting inequality,

$$n^2(m + n) > \text{ or } < m^2(3m - n).$$

But, $m > n$ (1)

Square (1), $m^2 > n^2$ (2)

Multiply (1) by 2, $2m > 2n$.

Add $m - n$, $3m - n > m + n$ (3)

Multiply the corresponding members of (2) and (3),

$$m^2(3m - n) > n^2(m + n).$$

Therefore, $4m^3(m - n) > m^4 - n^4$.

5. Find the sum of $x^2 + y > 1 - a$, $y^2 - 2a > b + 4$, $3x + y < 2a + 1$, and $y^2 - 3x^2 < 5 - a$.

6. From $a^2 + 2ax^2 < 5$ take $a(a + x^2) > n^2 - 1$.

7. From $a^2 < 3 - x^2$ subtract $2x^2 > b$.

Multiply:

8. $(a+b)^2 > (x-y)^2$ by -3; $3-y^3 < 5-x^3$ by x^3+y^3.

9. Divide $a^4 - b^4 > a^2 + b^2$ by $a^2 + b^2$.

10. Divide $11a^2 + 88b > 121n^2$ by -11.

Perform the indicated operations and simplify:

11. $(m-1<5)(m+1<10)$; $(a<n+b)(n-b>c)$.

12. $(-2>-3)^3$; $(5>2)\div(3<4)$; $\sqrt{25>9}$.

13. $[-243>-1024]^{\frac{1}{5}}$; $(n+1)^3>n^3-n^2+4n$.

14. $m^3-n^3>(m-n)(m^2+n^2)$; $\sqrt[3]{-64<8}$.

15. $(m^2-n^2<x^2)\div(\frac{1}{2}x>m+n)$; $[-n>y]^3$.

Solve:

16. $(x-2)^2>x^2+6x-25$; $\sqrt{(x-1)^2+3x^2+6}>x^{\frac{3}{2}}$.

17. $x-2>\sqrt{\dfrac{3x^2-4}{3}}$; $\sqrt{3}-4\sqrt{x}>\sqrt{16x-5}$.

18. $\begin{cases} 3y+2x>3. \\ 4>4y+x. \end{cases}$

19. $\begin{cases} x+\frac{3}{2}>\sqrt{x^2-3x+y}. \\ 5>x-y. \end{cases}$

20. $\begin{cases} 4y-x>y+4. \\ 3x-6y>1-4y. \end{cases}$

21. $\begin{cases} 3x-1>x+3y. \\ 2y-3x^2=3x-3x^2. \end{cases}$

22. $38x-7-15x^2<0$; $6x^2+7x+2<0$.

23. $17x-6x^2-5<0$; $6x+11-x^2<2x-10$.

Find integral values of x in the following:

24. $\begin{cases} 3\frac{1}{3}x-.5x>5. \\ 2.5x+\frac{1}{3}x<8. \end{cases}$

25. $\begin{cases} x+7<15. \\ 2x+10>20. \end{cases}$

26. $\begin{cases} \frac{1}{2}x-\frac{1}{3}x<3. \\ 7x-15>4x+30. \end{cases}$

27. $\begin{cases} 2x-5>31. \\ 3x-20<2x. \end{cases}$

28. $x^2+2x-15<0$; $x^2+16x+63>0$.

Show that:

29. $\sqrt{19}+\sqrt{3} > \sqrt{10}+\sqrt{7}$; $\sqrt{5}+\sqrt{14} > \sqrt{3}+3\sqrt{2}$.

If $a > b$, show that:

30. $\sqrt{a^2 - b^2} + \sqrt{a^2 - (a - b)^2} > a$.

31. $a^3 - b^3 < 3\,a^2\,(a - b)$ and $> 3\,b^2\,(a - b)$.

32. $a - b > \dfrac{a^4 - b^4}{4\,a^3}$ and $< \dfrac{a^4 - b^4}{4\,b^3}$.

If $x^2 = a^2 + b^2$, $y^2 = c^2 + d^2$, show that:

33. $x\,y > a\,c + b\,d$ or $a\,d + b\,c$.

Show that:

34. $(a\,b + x\,y)\,(a\,x + b\,y) > 4\,a\,b\,x\,y$.

35. $(a + b)\,(a + c)\,(b + c) > 8\,a\,b\,c$.

36. Show that the sum of any fraction and its reciprocal is greater than 2.

37. In how many ways may a street 20 yards long and 15 wide be paved with two kinds of stones; one kind being $3\frac{2}{3}$ feet long and 3 wide, the other $4\frac{3}{4}$ feet long and 4 wide?

38. A and B set out at the same time to meet each other; on meeting it appeared that A had travelled a miles more than B, and that A could have gone B's distance in n hours, and B could have gone A's distance in m hours. Find the distance between the two places. Solve when $a = 18$, $n = 378$, and $m = 672$.

CHAPTER XXVI.

SERIES.

153. A **Series** is an expression in which the successive terms are formed according to some fixed law;

As, 1, 2, 4, 8, ..., in which each term is double the preceding term; a, $a + d$, $a + 2d$, $a + 3d$, ..., in which each term exceeds the preceding term by d.

ARITHMETICAL PROGRESSION.

154. The expressions 1, 5, 9, 13, 17, ..., and 15, 10, 5, 0, -5, -10, ..., are called arithmetical progressions or series. The first is an **increasing series**, and the second a **decreasing series.** The general form for such a series is,

$$a,\ a + d,\ a + 2d,\ a + 3d,\ a + 4d,\ a + 5d,\ a + 6d,\ \ldots$$

in which a is the first term and d the common difference; the series will be *increasing* or *decreasing* according as d is positive or negative. Hence,

An **Arithmetical Progression** is a series in which the adjacent terms increase or decrease by a common difference.

In every arithmetical series the following elements occur, any three of which being given, the other two may be found:

The first term, or a.
The last term, or l.
The common difference, or d.
The number of terms, or n.
The sum of the terms, or s.

By an examination of the general form it is seen that the coefficient of d is always 1 *less than the number of the term.*

Thus, the 2d term is $a + d$, or $a + (2 - 1)d$,

3d term is $a + 2d$, or $a + (3 - 1)d$,

4th term is $a + 3d$, or $a + (4 - 1)d$,

12th term is $a + 11d$, or $a + (12 - 1)d$, and so on.

In the nth, or last term, the coefficient of d is $n - 1$. Hence,

To Find the Last Term of an Arithmetical Series, *when the first term, the common difference, and the number of terms are given.*

$$l = a + (n - 1)d \tag{i}$$

Note. The common difference may always be found by subtracting any term of the series from that which immediately follows it.

Example 1. Find the 18th term of the series $\frac{7}{6}$, $\frac{4}{3}$, $\frac{3}{2}$, etc.

Process. Here, $n = 18$, $a = \frac{7}{6}$, and $d = \frac{4}{3} - \frac{7}{6} = \frac{1}{6}$.
Substitute these values in (i), $l = \frac{7}{6} + (18 - 1)\frac{1}{6} = 4$.

Example 2. Find the 30th term of the series $x + y$, x, $x - y$, etc.

Process. Here, $n = 30$, $d = x - (x + y) = -y$, and $a = x + y$.
Substitute these values in (i), $l = x + y + (30 - 1)(-y) = x - 28y$.

Exercise 135.

Find:

1. The 15th term of 7, 3, -1, ...
2. The 27th and 41st terms of 5, 11, 17, ...
3. The 20th and 13th terms of -3, -2, -1, ...
4. The 37th and 89th terms of -2.8, 0, 2.8, ...
5. The 40th term of $2a - b$, $4a - 3b$, $6a - 5b$, ...
6. The 15th and 8th terms of $\frac{7}{6}$, 1, $\frac{5}{6}$, ...

7. The first term is $\frac{7}{6}$, the 102d is 18. Find the common difference.

8. The 21st term is 53, and the common difference is $-2\frac{1}{2}$. Find the first term.

9. The first term is $5\frac{1}{3}$, and the common difference is $3\frac{1}{3}$. What term will be 42?

10. The first term is $\frac{1}{5}$, the common difference is $\frac{24}{25}$, and the last term is $17\frac{12}{25}$. Find the number of terms.

11. The 54th and 4th terms are -125 and 0. Find the 42d term.

12. Find three terms whose common difference is $\frac{1}{4}$, such that the product of the second and third exceeds that of the first and second by $1\frac{1}{8}$.

155. Taking the elements as given in Art. 154:

$$s = a+(a+d)+(a+2d)+(a+3d)+(a+4d)+\dots l,$$

or $$s = l+(l-d)+(l-2d)+(l-3d)+(l-4d)+\dots a.$$

Add, $2s = (a+l)+(a+l)+(a+l)+(a+l)+(a+l)+\dots$ to n terms,

or $$2s = n(a+l) \tag{1}$$

Substitute the value of l from (i) (Art. 154) in (1),

$2s = n[2a+(n-1)d]$. Hence (solve for s),

To Find the Sum of all the Terms of an Arithmetical Series.

$$s = \frac{n}{2}(a+l) \tag{ii}$$

$$s = \frac{n}{2}[2a+(n-1)d] \tag{iii}$$

Example 1. Find the sum of an arithmetical series of 17 terms, the first term being $5\frac{1}{2}$, and the last term $25\frac{1}{2}$.

Process. Here, $n = 17$, $a = 5\frac{1}{2}$, and $l = 25\frac{1}{2}$.
Substitute these values in (ii), $s = \frac{17}{2}(5\frac{1}{2}+25\frac{1}{2}) = 263\frac{1}{2}$.

EXAMPLE 2. Find the sum of the series $3\frac{1}{2}, 1, -1\frac{1}{2}, \ldots,$ to 19 terms.

Process. Here, $n = 19$, $a = 3\frac{1}{2}$, and $d = 1 - 3\frac{1}{2} = -2\frac{1}{2}$.
Substitute these values in (iii),

$$s = \tfrac{19}{2}[2 \times 3\tfrac{1}{2} + (19-1)(-2\tfrac{1}{2})] = -361.$$

EXAMPLE 3. Find the sum of $m - \frac{1}{m}, 3m - \frac{2}{m}, 5m - \frac{3}{m}, \ldots,$ to m terms.

Process. Here, $n = m$, $a = m - \frac{1}{m}$, and $d = 3m - \frac{2}{m} - \left(m - \frac{1}{m}\right) = 2m - \frac{1}{m}$.

Substitute in (iii),

$$s = \frac{m}{2}\left[2\left(m - \frac{1}{m}\right) + (m-1)\left(2m - \frac{1}{m}\right)\right] = \frac{2m^3 - m - 1}{2}.$$

EXAMPLE 4. The first term of a series is $3m$, the last $-35m$, and the sum $-320m$. Find the number of terms and the common difference.

Process. Here, $s = -320m$, $a = 3m$, and $l = -35m$.
Substitute in (ii),

$$-320m = \frac{n}{2}(3m - 35m) = -16mn. \qquad \therefore n = 20.$$

Substitute in (iii),

$$-320m = \tfrac{20}{2}[6m + 19d] = 60m + 190d. \quad \therefore d = -2m.$$

EXAMPLE 5. How many terms of the series $-6\frac{4}{5}, -6\frac{2}{5}, -6, \ldots,$ must be taken to make $-52\frac{4}{5}$?

Process. Here, $s = -52\frac{4}{5}$, $a = -6\frac{4}{5}$, and $d = \frac{2}{5}$.

Substitute in (iii), $-52\frac{4}{5} = \frac{n}{2}[-\frac{68}{5} + (n-1) \times \frac{2}{5}]$.

Simplify and solve for n, $n = 11$ or 24.

Query. Do both of these values satisfy the conditions? In explanation write out 24 terms of the series and observe that the last 13 terms destroy each other.

Exercise 136.

Find the sum of:

1. 5, 9, 13, ..., to 19 terms.

2. $10\frac{1}{2}$, 9, $7\frac{1}{2}$, ..., to 94 terms.

3. $3a$, a, $-a$, ..., to a terms.

4. $3\frac{1}{3}$, $2\frac{1}{2}$, $1\frac{2}{3}$, ..., to n terms.

5. $\frac{m-1}{m}$, $\frac{m-2}{m}$, $\frac{m-3}{m}$, ..., to m terms.

6. $\frac{2a^2-1}{a}$, $4a-\frac{3}{a}$, $\frac{6a^2-5}{a}$, ..., to n terms.

7. a, $\frac{4a+b}{3}$, $\frac{5a+2b}{3}$, ..., to 19 terms.

8. The first term is $3\frac{6}{7}$, and the sum of 14 terms is $84\frac{1}{3}$. Find the last term.

9. The sum of 40 terms is 0, and the common difference is $-\frac{2}{3}$. Find the first term.

10. Find the number of terms and common difference: (1) when the sum is 24, the first term 9, and the last -6; (2) the sum $49a$, the first term a, and the last $13a$.

11. The sum of 12 terms is 150, and the first is $5\frac{1}{6}$. Determine the series.

12. Show that the sum of the first n odd numbers is n^2.

13. Find the sum of all the odd numbers between 100 and 200.

14. The sum of five terms is 15, and the difference of the squares of the extremes is 96. Find the terms.

15. Find the sum of $\frac{1}{1+\sqrt{x}}, \frac{1}{1-x}, \frac{1}{1-\sqrt{x}}, \ldots,$ to n terms.

156. a is called the arithmetical mean between $a-d$ and $a+d$. Hence,

An **Arithmetical Mean** is the middle term of three numbers in arithmetical series.

If a and b represent two numbers, and A their arithmetical mean, the common difference is $A-a$, or $b-A$. Therefore,

$$A-a=b-A. \quad \text{Hence (solve for } A\text{)},$$

To Find the Arithmetical Mean Between two Terms.

$$\boldsymbol{A=\frac{a+b}{2}} \qquad \text{(iv)}$$

If a and l represent any two numbers, and m the *number* of means between them, the whole number of terms is $m+2$, or $m+2=n$.

Substitute this value for n in (i) (Art. 154),

$$l=a+(m+1)\,d. \quad \text{Hence (solve for } d\text{)},$$

To Insert any Number of Arithmetical Means Between two Terms.

$$\boldsymbol{d=\frac{l-a}{m+1}} \qquad \text{(v)}$$

This finds d, and the m required means are,

$$\boldsymbol{a+d,\ a+2\,d,\ a+3\,d,\ a+4\,d,\ \ldots,\ a+m\,d.}$$

EXAMPLE 1. Find the arithmetical mean between: (1) 27 and -5; (2) m^2+mn-n^2 and m^2-mn+n^2.

Process. (1) Here, $a=27$, $b=-5$.

Substitute in (iv), $A=\frac{27-5}{2}=11.$

(2) Here, $a=m^2+mn-n^2$, $b=m^2-mn+n^2$.

Substitute in (iv), $A=\frac{m^2+mn-n^2+m^2-mn+n^2}{2}=m^2.$

EXAMPLE 2. Insert five arithmetical means between 12 and 20.

Process. Here, $a = 12$, $l = 20$, and $m = 5$.

Substitute in (v), $d = \dfrac{20 - 12}{5 + 1} = 1\frac{1}{3}$.

The series is 12, $13\frac{1}{3}$, $14\frac{2}{3}$, 16, $17\frac{1}{3}$, $18\frac{2}{3}$, 20.

Exercise 137.

Insert:

1. 14 arithmetical means between $-7\frac{1}{5}$ and $-2\frac{1}{5}$.

2. 16 arithmetical means between 7.2 and -6.4.

3. 10 arithmetical means between $5m-6n$ and $5n-6m$.

4. 4 arithmetical means between -1 and -7.

5. x arithmetical means between x^2 and 1.

6. Find the arithmetical mean between $\dfrac{m-n}{m+n}$ and $\dfrac{m+n}{m-n}$.

7. The arithmetical mean between two numbers is -9, and the mean between four times the first and twelve times the second is -66. Find the numbers.

GEOMETRICAL PROGRESSION.

157. The expressions 3, 9, 27, 81, ..., and $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots,$ are called geometrical progressions or series. The general form for such a series is,

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ ar^6,\ ar^7,\ \ldots,$$

In which a is the first term, and r a constant factor or ratio. Hence,

A **Geometrical Progression** is a series in which the adjacent terms increase or decrease by a constant factor.

The **Common Ratio** is the factor by which each term is multiplied to form the next one.

In every geometrical series the following elements occur; any three of which being given, the other two may be found.

The first term, or a.
The last term, or l.
The common ratio, or r.
The number of terms, or n.
The sum of the terms, or s.

By an examination of the general form it is seen that the exponent of r is always 1 *less than the number of the term.*

Thus, the 2d term is ar,
3d term is ar^2,
4th term is ar^3,
12th term is ar^{11}, and so on.

In the nth, or last term, the exponent of r is $n-1$. Hence,

To Find the Last Term of a Geometrical Series, *when the first term, the common ratio, and the number of terms are given.*

$$l = ar^{n-1} \qquad \text{(i)}$$

Notes: 1. The common ratio is found by dividing any term by that which immediately precedes it.

2. A geometrical series is said to be increasing or decreasing, according as the common ratio is greater than 1, or less than 1.

3. An arithmetical series is formed by repeated addition or subtraction; a geometrical series by repeated multiplication.

Example 1. Find the 8th term of the series .008, .04, .2, etc.

Process. Here, $a = .008$, $n = 8$, and $r = .04 \div .008 = 5$.
Substitute in (i), $l = .008 \times 5^{8-1} = 625$.

Example 2. Find the 10th term of $\frac{x^2}{y}$, x, y, $\frac{y^2}{x}$, ...

Process. Here, $a = \frac{x^2}{y}$, $n = 10$, and $r = x \div \frac{x^2}{y} = \frac{y}{x}$.

Substitute in (i), $l = \frac{x^2}{y}\left(\frac{y}{x}\right)^9 = x^{-7}y^8$.

Exercise 138.

Find:

1. The 5th and 8th terms of 3, 6, 12, ...
2. The 10th and 16th terms of 256, 128, 64, ...
3. The 8th and 12th terms of 81, -27, 9, ...
4. The 14th and 7th terms of $\frac{1}{64}$, $\frac{1}{32}$, $\frac{1}{16}$, ...
5. The 6th term of $\frac{x}{y}$, $\frac{mx}{y^2}$, $\frac{m^2x}{y^3}$, ...
6. The mth term of x, x^3, x^5, ...

7. The 3d and 6th terms are $\frac{9}{16}$ and $-\frac{9}{2}$. Find the series and the 12th term.

8. The 5th and 9th terms are $\frac{27}{16}$ and $\frac{1}{3}$. Find the series.

9. If from a line a inches in length, one third be cut off, then one third of the remainder, and so on; what part of it will remain when this has been done 5 times? When t times.

158. Taking the elements as given in Art. 157,

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} \tag{1}$$

Multiply (1) by r,

$$sr = ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar \tag{2}$$

Subtract (1) from (2),

$$sr - s = ar^n - a \tag{3}$$

Substitute the value of ar^n from (i) (Art. 157) in (3), and factor the result, $s(r-1) = rl - a$. Hence (solve for s),

To Find the Sum of all the Terms of a Geometrical Series.

$$\boldsymbol{s = \frac{a(r^n - 1)}{r - 1}} \qquad \text{(ii)}$$

$$\boldsymbol{s = \frac{rl - a}{r - 1}} \qquad \text{(iii)}$$

EXAMPLE 1. Find the 6th term and the sum of $-\frac{1}{3}, \frac{1}{2}, -\frac{3}{4}, \ldots.$

Process. Here, $a = -\frac{1}{3}$, $n = 6$, and $r = -\frac{3}{2}$.

Substitute in (i) (Art. 157), $l = -\frac{1}{3} \times (-\frac{3}{2})^5 = \frac{81}{32}$.

Substitute in (iii), $$s = \frac{-\frac{3}{2} \times \frac{81}{32} + \frac{1}{3}}{-\frac{3}{2} - 1} = \frac{133}{96}.$$

EXAMPLE 2. Find the least term and the sum of $3, -9, 27, \ldots,$ to 7 terms.

Process. Here, $a = 3$, $n = 7$, and $r = -3$.

Substitute in (i) (Art. 157), $l = 3(-3)^6 = 2187$.

Substitute in (ii), $$s = \frac{3[(-3)^7 - 1]}{-3 - 1} = 1641.$$

Exercise 139.

Find the sum of:

1. $3, -1, \frac{1}{3}, \ldots,$ to 6 terms.

2. $-\frac{2}{5}, \frac{1}{2}, -\frac{5}{8}, \ldots,$ to 6 terms.

3. $\frac{2}{3}, -\frac{1}{6}, \frac{1}{24}, \ldots,$ to 8 terms.

4. $\frac{1}{\sqrt{3}}, 1, \frac{3}{\sqrt{3}}, \ldots,$ to 8 terms.

5. $1, 3, 3^2, \ldots,$ to m terms.

6. $2, -4, 8, \ldots,$ to $2m$ terms.

7. The 7th and 4th terms are 625 and -5. Find the 1st term, and the sum of the 4th to the 7th terms inclusive.

8. The sum of the first 10 terms is equal to 33 times the sum of the first 5 terms. Find the common ratio.

9. The sum of three numbers in geometrical progression is 215, and the first term is 5. Find the common ratio and the numbers.

159. A **Geometrical Mean** is the middle term of three numbers in geometrical series.

If a and b represent two numbers, and G their geometrical mean, the common ratio is $\frac{G}{a}$, or $\frac{b}{G}$. Therefore,

$$\frac{G}{a} = \frac{b}{G}. \text{ Hence (solve for } G\text{),}$$

To Find the Geometrical Mean Between two Terms

$$G = \sqrt{a\,b} \qquad \text{(iv)}$$

If a and b represent any two numbers, and m the *number* of means between them, the whole number of terms is $m + 2$, or $m + 2 = n$.

Substitute this value for n in (i) (Art. 157),

$$l = a\,r^{m+1}. \text{ Hence (solve for } r\text{),}$$

To Insert any Number of Geometrical Means Between two Terms.

$$r = \sqrt[m+1]{\frac{l}{a}} \qquad \text{(v)}$$

This finds r, and the m required means are,

$$a\,r,\ a\,r^2,\ a\,r^3,\ a\,r^4,\ a\,r^5,\ \ldots,\ a\,r^m.$$

EXAMPLE 1. Find the geometrical mean between: (1) $\frac{1}{\sqrt{3}}$ and $\frac{3}{\sqrt{3}}$; (2) $3\,x^3\,y$ and $12\,x\,y^3\,z$.

Process. (1) Here, $a = \frac{1}{\sqrt{3}}$, and $b = \frac{3}{\sqrt{3}}$.

Substitute in (iv), $G = \sqrt{\frac{1}{\sqrt{3}} \times \frac{3}{\sqrt{3}}} = 1.$

(2) Here, $a = 3\,x^3\,y$ and $b = 12\,x\,y^3\,z$.

Substitute in (iv), $G = \sqrt{3\,x^3\,y \times 12\,x\,y^3\,z} = 6\,x^2\,y^2\sqrt{z}.$

EXAMPLE 2. Insert six geometrical means between 14 and $-\frac{7}{64}$.

Process. Here, $a = 14$, $l = -\frac{7}{64}$, and $m = 6$.

Substitute in (v), $r = \sqrt[7]{-\frac{1}{128}} = -\frac{1}{2}$.

Hence, the series is $14, -7, \frac{7}{2}, -\frac{7}{4}, \frac{7}{8}, -\frac{7}{16}, \frac{7}{32}, -\frac{7}{64}$.

Exercise 140.

Find the geometrical mean between:

1. 7 and 252; a^3b and ab^3; $\frac{2}{3}$ and $\frac{81}{16}$; $\frac{5}{4}$ and $\frac{45}{64}$.

2. $\frac{1}{10}$ and $\frac{1}{1000}$; $4x^2 - 12x + 9$ and $9x^2 + 12x + 4$.

Insert:

3. 2 geometrical means between 5 and 320.

4. 2 geometrical means between 1 and $\frac{1}{8}$.

5. 3 geometrical means between 100 and $2\frac{14}{25}$.

6. 6 geometrical means between 14 and $-\frac{7}{64}$.

7. 7 geometrical means between 2 and 13,122.

8. Which is the greater, and how much greater, the arithmetical or geometrical mean between 1 and $\frac{1}{9}$.

9. Find two numbers whose sum is 10, and whose geometrical mean is 4.

HARMONICAL PROGRESSION.

160. The expressions $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$, and $4, -\frac{4}{3}, -\frac{4}{5}, -\frac{4}{7}, \ldots$, are called harmonical progressions or series, because their reciprocals $1, 3, 5, 7, \ldots$, and $\frac{1}{4}, -\frac{3}{4}, -\frac{5}{4}, -\frac{7}{4}, \ldots$, form arithmetical series. The general form for such a series is,

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots, \frac{1}{a+(n-1)d}. \quad \text{Hence,}$$

An **Harmonical Progression** is a series the reciprocals of whose terms form an arithmetical series.

Notes: 1. Evidently all questions relating to harmonical progression are readily solved by writing the reciprocals of the terms so as to form an arithmetical series.

2. There is no general formula for finding the sum of the terms of a harmonical series.

3. The term **harmonical** is derived from the fact that musical strings of equal thickness and tension produce *harmony* when sounded together, if their lengths are represented by the *reciprocals* of the series of natural numbers; that is, by the series $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, etc. Harmonical properties are also interesting because of their importance in geometry.

EXAMPLE 1. Find the mth term of the series $3, 1\frac{1}{2}, 1, \frac{3}{4}, \frac{3}{7}$, etc.

Solution. Taking the reciprocals of the terms, we have $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{7}{3}$, etc.; an arithmetical series.

Here, $a = \frac{1}{3}$, $d = \frac{1}{3}$, and $n = m$.

Substituting in (i) (Art. 154), $d = \frac{1}{3} + (m - 1)\frac{1}{3} = \frac{m}{3}$. Taking the reciprocal of this value for the required term, we have $\frac{3}{m}$.

EXAMPLE 2. The 12th term is $\frac{1}{5}$, and the 19th term is $\frac{3}{22}$. Find the series.

Process. The 12th and 19th terms of the corresponding arithmetical series are 5 and $\frac{22}{3}$.

From (i) (Art. 154), $5 = a + 11\,d$,

$$\tfrac{22}{3} = a + 18\,d,$$

Solving for a and d, $a = \frac{4}{3}$ and $d = \frac{1}{3}$.

The arithmetical series is, $\frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, 3, \frac{10}{3}, \ldots$

The harmonical series is, $\frac{3}{4}, \frac{3}{5}, \frac{1}{2}, \frac{3}{7}, \frac{3}{8}, \frac{1}{3}, \frac{3}{10}, \ldots$

161. A **Harmonical Mean** is the middle term of three numbers in harmonical series.

If a and b represent two numbers, and H their harmonical mean, the corresponding arithmetical series is $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$. The common difference is $\frac{1}{H} - \frac{1}{a}$, or $\frac{1}{b} - \frac{1}{H}$. Therefore,

$$\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}. \quad \text{Hence (solve for } H\text{)},$$

To Find the Harmonical Mean Between two Numbers.

$$H = \frac{2ab}{a+b} \tag{i}$$

EXAMPLE 1. Find the harmonical mean between: (1) $\frac{1}{4}$ and $\frac{1}{10}$; (2) $x+y$ and $x-y$.

Process. (1) Here, $a = \frac{1}{4}$ and $b = \frac{1}{10}$.

Substitute in (i), $H = \frac{1}{7}$

(2) Here, $a = x+y$ and $b = x-y$.

Substitute in (i), $H = \dfrac{x^2 - y^2}{x}$.

EXAMPLE 2. Insert three harmonical means between $\frac{3}{4}$ and $\frac{3}{10}$.

Process. The terms of the corresponding arithmetical series are $\frac{4}{3}$ and $\frac{10}{3}$.

Here, $a = \frac{4}{3}$, $l = \frac{10}{3}$, and $m = 3$.

Substitute in (v) (Art. 156), $d = \frac{1}{2}$.

The three arithmetical means are $\frac{11}{6}, \frac{14}{6}, \frac{17}{6}$.

The required harmonical means are $\frac{6}{11}, \frac{3}{7}, \frac{6}{17}$.

Exercise 141.

1. Find the 8th term of $1\frac{1}{3}, 1\frac{11}{17}, 2\frac{2}{13}, \ldots$.

2. Find the 21st term of $2\frac{1}{2}, 1\frac{12}{13}, 1\frac{9}{16}, \ldots$.

3. The 39th term is $\frac{1}{11}$, and the 54th term is $\frac{1}{26}$. Find the series.

4. The 2d term is 2, and the 3d term is $\frac{4}{3}$. Find the first six terms.

Insert:

5. One harmonical mean between 1 and 13.

6. 3 harmonical means between $2\frac{2}{5}$ and 12.

7. 4 harmonical means between $\frac{2}{3}$ and $\frac{2}{13}$.

8. 6 harmonical means between 3 and $\frac{6}{23}$.

9. The arithmetical mean of two numbers is 9, and the harmonical mean is 8. Find the numbers.

10. The difference of the arithmetical and harmonical means between two numbers is 1. Find the numbers; one being three times the other.

11. Find two numbers such that the sum of their arithmetical, geometrical, and harmonical means is $9\frac{4}{5}$, and the product of these means is 27.

12. The arithmetical mean between two numbers exceeds the geometrical by $2\frac{1}{2}$, and the geometrical exceeds the harmonical by 2. Find the numbers.

13. The sum of three terms of a harmonical series is 37, and the sum of their squares is 469. Find the numbers.

14. The sum of three consecutive terms in harmonical series is $1\frac{1}{12}$, and the first term is $\frac{1}{2}$. Find the numbers.

15. Arrange the arithmetical, geometrical, and harmonical means between two numbers a and b in order of magnitude.

16. If 50 potatoes are placed in a line 3 feet from each other, and the first is 3 feet from a basket, how far will a person travel, starting from the basket, to gather them up singly, and return with each to the basket?

17. There are four numbers in geometrical progression, the first of which is less than the fourth by 21, and the difference of the extremes divided by the difference of the means is equal to $3\frac{1}{2}$. Find the numbers.

CHAPTER XXVII.

RATIO AND PROPORTION.

162. THE **Ratio** of two numbers is their relative magnitude, and is expressed by the fraction of which the first is the numerator and the second the denominator.

Thus, the ratio of 10 to 5 is expressed by the fraction $\frac{10}{5}$; the ratio of $\frac{3}{4}$ to $\frac{5}{6}$ is expressed by the fraction $\frac{3}{4} \div \frac{5}{6}$ $(= \frac{9}{10})$.

The ratio of two quantities of the same kind is equal to the ratio of the two numbers by which they are expressed.

Thus, the ratio of \$5 to \$6 is $\frac{5}{6}$; of 15 apples to 3 apples is $\frac{15}{3}$; of $3\frac{3}{4}$ feet to $5\frac{1}{3}$ feet is $3\frac{3}{4} \div 5\frac{1}{3} = \frac{45}{64}$.

The **Sign** of ratio is the colon :, $\div$, or the fractional form of indicating division.

Thus, the ratio of a to b is expressed by $a : b$, or $a \div b$, or $\frac{a}{b}$, any one of which may be read "a is to b," or "ratio of a to b."

The **Terms** of a ratio are the numbers compared. The first term is called the **antecedent**, the second the **consequent**, and the two terms together are called a **couplet**.

A ratio is called a ratio of **greater inequality**, of **less inequality**, or of **equality**, according as the antecedent is **greater** than, **less** than, or **equal** to, the consequent.

An **Inverse Ratio** is one in which the terms are interchanged; as, the ratio of $7 : 8$ is the inverse of the ratio $8 : 7$.

A **Compound Ratio** is the product of two or more simple ratios; as, the compound ratio $2 : 3$, $5 : 4$, $15 : 6$, is $150 : 72$.

Notes: 1. A quantity may be defined as a definite portion of any magnitude. Thus, any definite number of dollars, pounds, bushels, acres, feet, yards, or miles, is a quantity.

2. To compare two quantities they must be expressed in terms of the same unit. Thus, the ratio of 2 rods to 9 inches is expressed by the fraction,

$$\frac{16\frac{1}{2} \times 2 \times 12}{9} = \frac{396}{9}.$$

163. Evidently the ratios $4 : 5$, $8 : 10$, $\frac{20}{5} : \frac{25}{5}$, are equal to each other. In general,

$$\frac{a}{b} = \frac{ma}{mb}. \quad \text{Hence,}$$

I. *If the terms of a ratio are multiplied or divided by the same number, the value of the ratio is not changed.*

The ratio $9 : 7$ is compared with the ratio $4 : 3$ by comparing $\frac{9}{7}$ and $\frac{4}{3}$. $\frac{9}{7} = \frac{27}{21}$, and $\frac{4}{3} = \frac{28}{21}$. Therefore, $4 : 3$ is greater than $9 : 7$. Hence,

II. *Ratios are compared by comparing the fractions that represent them.*

If to each term of the ratio $5 : 4$ we add 16, the new ratio, $21 : 20$, is less than the ratio $5 : 4$, because $\frac{5}{4}$ is greater than $\frac{21}{20}$. If to each term of the ratio $4 : 5$ we add 16, the new ratio, $20 : 21$, is greater than the ratio $4 : 5$. Hence,

III. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same number to both its terms.*

If from each term of the ratio $32 : 30$ we subtract 24, the new ratio, $8 : 6$, is greater than the ratio $32 : 30$. If from each term of the ratio $28 : 30$ we subtract 15, the new ratio, $13 : 15$, is less than the ratio $28 : 30$. Hence,

IV. *A ratio of greater inequality is increased, and a ratio of less inequality is diminished, by taking the same number from both terms.*

Suppose $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=r.$

Simplify, $b\,r=a,\ d\,r=c,\ f\,r=e,\ h\,r=g.$

Add the corresponding members and factor the result,

$$(b+d+f+h)\,r=a+c+e+g.$$

Therefore, $r=\frac{a+c+e+g}{b+d+f+h}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}.$ Hence,

V. *In a series of equal ratios, the sum of the antecedents divided by the sum of the consequents is equal to any antecedent divided by its consequent.*

Notes: 1. The sign :, is an exact equivalent for the sign of division; and is a modification of ÷.

2. A **Duplicate Ratio** is the ratio of the squares; a **Triplicate**, of the cubes; a **Subduplicate**, of the square roots; a **Subtriplicate**, of the cube roots of two numbers. Thus, $a^2:b^2$; $a^3:b^3$; $\sqrt{a}:\sqrt{b}$; $\sqrt[3]{a}:\sqrt[3]{b}$ are respectively the duplicate, triplicate, subduplicate, and subtriplicate ratios of a to b.

EXAMPLE 1. Find the ratio compounded of the duplicate ratio of $\frac{2a}{b}:\frac{a^2}{b^2}\sqrt{6}$, and the ratio $3\,a\,x:2\,b\,y$.

Process. The duplicate ratio of $\frac{2a}{b}:\frac{a^2}{b^2}\sqrt{6}$ is $\frac{4a^2}{b^2}:\frac{6a^4}{b^4}$.

The compound ratio $\frac{4a^2}{b^2}:\frac{6a^4}{b^4}$, $3\,a\,x:2\,b\,y$, is $\frac{12a^3x}{b^2}:\frac{12a^4by}{b^4}$.

But $\frac{12a^3x}{b^2}:\frac{12a^4by}{b^4}=\frac{12a^3x}{b^2}\div\frac{12a^4by}{b^4}=\frac{bx}{ay}=b\,x:a\,y.$

EXAMPLE 2. If $15\,(2\,x^2-y^2)=7\,x\,y$, find the ratio $x:y$.

Process. From the given equation, $x^2-\frac{7}{30}\,x\,y=\frac{1}{2}\,y^2$.

Complete the square and solve for x, $x=\frac{5}{6}\,y$, or $-\frac{3}{5}\,y$.

Therefore, $\frac{x}{y}=\frac{5}{6}$, or $-\frac{3}{5}$.

Exercise 142.

Find the ratio compounded of:

1. The ratio $2\,a:3\,b$, and the duplicate ratio of $9\,b^2:a\,b$.
2. The subduplicate ratio of $64:9$, and the ratio $27:56$.

3. The duplicate ratio of $4 : 15$, and the triplicate ratio of $5 : 2$.

4. $1 - x^2 : 1 + y$, $x - xy^2 : 1 + x^2$, and $1 : x - x^2$.

5. $\frac{a+b}{a-b}$, $\frac{a^2+b^2}{(a+b)^2}$, $\frac{(a^2-b^2)^2}{a^4-b^4}$, $\frac{a^2-9a+20}{a^2-6a}$, and $\frac{a^2-13a+42}{a^2-5a}$.

Simplify each of the ratios:

6. $5ax : 4x$; $16xy : 20x^2$; $2x^2y : \frac{1}{4}x^3$.

7. $\frac{7}{6}axy : \frac{5}{24}ay^2$; $\frac{n(n-1)ax^3}{2} : a^2nx^3$.

Arrange the following ratios in order of magnitude:

8. $5 : 6$, $7 : 8$, $41 : 48$, and $31 : 36$.

9. $a - b : a + b$, and $a^2 - b^2 : a^2 + b^2$, when $a > b$.

10. For what value of x will the ratio $15 + x : 17 + x$ be equal to the ratio $1 : 12$?

11. Find $x : y$, if $x^2 + 6y^2 = 5xy$.

12. Find the ratio of x to y, if the ratio $4x + 5y : 3x - y$ is equal to 2.

13. What number must be added to each term of the ratio $a : b$, that it may become equal to the ratio $m : n$?

14. What number must be subtracted from the consequent of the ratio $a : b$, that it may become equal to the ratio $m : n$?

15. A certain ratio will be equal to $2 : 3$, if 2 be added to each of its terms; and it will be equal to $1 : 2$, if 1 be subtracted from each of its terms. Find the ratio.

16. If $a : b$ be in the duplicate ratio of $a + x : b + x$, find x.

17. Show that a duplicate ratio is greater or less than its simple ratio, according as it is a ratio of greater or less inequality.

PROPORTION.

164. A **Proportion** is an equality of ratios. Four numbers are in proportion, when the first divided by the second is equal to the third divided by the fourth.

Thus, if $\frac{a}{b} = \frac{c}{d}$, then a, b, c, d, are called *proportionals*, or are said to be in *proportion*, and they may be written in either of the forms:

$$a : b :: c : d,$$

read, "a is to b as c is to d;"

or $$a : b = c : d,$$

read, "the ratio of a to b is equal to the ratio of c to d;"

or $$\frac{a}{b} = \frac{c}{d},$$

read, "a divided by b equals c divided by d."

The **Terms** of a proportion are the four numbers compared. The first and third terms are called the **antecedents**, the second and fourth terms, the **consequents**; the first and fourth terms are called the **extremes**, the second and third terms, the **means**.

Thus, in the above proportion, a and c are the antecedents, b and d the consequents, a and d the extremes, b and c the means.

Note 1. The algebraic test of a proportion is that the two fractions which represent the ratios shall be equal.

Let $a : b :: c : d.$

By definition, $\frac{a}{b} = \frac{c}{d}.$

Free from fractions, $a d = b c.$ Hence,

I. *In any proportion the product of the extremes is equal to the product of the means.*

Note 2. If any three terms in a proportion are given, the fourth may be found from the relation that the product of the extremes is equal to the product of the means.

Let $a d = b c.$

Divide by $b d$, $\frac{a}{b} = \frac{c}{d}.$

By definition, $a : b :: c : d.$ Hence,

II. *If the product of two numbers is equal to the product of two others, either two may be made the extremes of a proportion and the other two the means.*

A **Mean Proportional** is a number used for both means of a proportion; as, b, in the proportion $a : b :: b : c$.

A **Third Proportional** is the fourth term of a proportion in which the means are equal; as, c, in the proportion $a : b :: b : c$

Let $a : b :: b : c.$
Therefore I., $b^2 = a c.$
Extract the square root, $b = \sqrt{a c}.$ Hence,

III. *A mean proportional between two numbers is equal to the square root of their product.*

Let $a : b :: c : d.$
Therefore I., $a d = b c.$

Divide by $c d$, $\frac{a}{c} = \frac{b}{d}.$

By definition, $a : c :: b : d.$ Hence,

IV. *If four numbers are in proportion, they will be in proportion by* **alternation**; *that is, the first will be to the third, as the second is to the fourth.*

Let $a : b :: c : d.$

Then I., $bc = ad.$

Divide by ac, $\frac{b}{a} = \frac{d}{c}.$

By definition, $b : a :: d : c.$ Hence,

V. *If four numbers are in proportion, they will be in proportion by* **inversion**; *that is, the second will be to the first as the fourth is to the third.*

Let $a : b :: c : d.$

By definition, $\frac{a}{b} = \frac{c}{d}.$

Add 1 to each member, $\frac{a}{b} + 1 = \frac{c}{d} + 1,$

or $\frac{a+b}{b} = \frac{c+d}{d}.$

Therefore, $a + b : b :: c + d : d.$ Hence,

VI. *If four numbers are in proportion, they will be in proportion by* **composition**; *that is, the sum of the first two will be to the second as the sum of the last two is to the fourth.*

Let $a : b :: c : d.$

By definition, $\frac{a}{b} = \frac{c}{d}.$

Subtract 1 from each member,

$$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

or $\frac{a-b}{b} = \frac{c-d}{d}.$

Therefore, $a - b : b :: c - d : d.$ Hence,

VII. *If four numbers are in proportion, they will be in proportion by* **division**; *that is, the difference of the first two will be to the second as the difference of the last two is to the fourth.*

Let $$a : b :: c : d.$$

Then VI., $$\frac{a+b}{b} = \frac{c+d}{c};$$

also VII., $$\frac{a-b}{b} = \frac{c-d}{c}.$$

Divide, $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

By definition, $a + b : a - b :: c + d : c - d$. Hence,

VIII. *If four numbers are in proportion, they will be in proportion by* **composition and division**; *that is, the sum of the first two will be to their difference as the sum of the last two is to their difference.*

Let
$$a : b :: c : d,$$
$$e : f :: g : h,$$
$$k : l :: m : n.$$

By definition, $\frac{a}{b} = \frac{c}{d}, \frac{e}{f} = \frac{g}{h}, \frac{k}{l} = \frac{m}{n}.$

Multiply the corresponding members of the equations together,

$$\frac{a\,e\,k}{b\,f\,l} = \frac{c\,g\,m}{d\,h\,n}.$$

By definition, $a\,e\,k : b\,f\,l :: c\,g\,m : d\,h\,n$. Hence,

IX. *The products of the corresponding terms of two or more proportions are in proportion.*

Let $$a : b :: c : d.$$

By definition, $$\frac{a}{b} = \frac{c}{d}.$$

Raise each member to the nth power, $$\frac{a^n}{b^n} = \frac{c^n}{d^n}.$$

Therefore, $a^n : b^n :: c^n : d^n.$

Extract the nth root of each member, $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \dfrac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}.$

Therefore, $a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}.$ Hence,

X. *In any proportion like powers or like roots of the terms are in proportion.*

A **Continued Proportion** is a series of equal ratios;

As, $8 : 4 :: 12 : 6 :: 10 : 5 :: 16 : 8$; $a : b :: c : d :: e : f :: g : h$, read "$a$ is to b as c is to d as e is to f as g is to h."

Note 3. Four numbers are said to form a continued proportion when each consequent is the antecedent of the next ratio; as, $a : b :: b : c :: c : d$.

Let $a : b :: c : d :: e : f :: g : h.$

By definition, $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \dfrac{g}{h}.$

By V., (Art. 163), $\dfrac{a+c+e+g}{b+d+f+h} = \dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \dfrac{g}{h}.$

Therefore, $a+c+e+g : b+d+f+h :: a : b.$ Hence,

XI. *In a continued proportion the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Example 1. If $\dfrac{a^2+b^2}{ab+bc} = \dfrac{ab+bc}{b^2+c^2}$, prove that b is a mean proportional between a and c.

Proof. Free the given equation from fractions, transpose and factor, $(b^2 - ac)^2 = 0$, or $b^2 = ac.$

Therefore II., $a : b :: b : c.$

Example 2. If $a : b :: c : d$, prove that $ma^2 + pb^2 + nab : mc^2 + pd^2 + ncd :: b^2 : d^2.$

Proof. From the given proportion VI., $\frac{a}{c} = \frac{b}{d}$ (1)

Multiply by $\frac{a}{c}$, $\frac{a^2}{c^2} = \frac{a\,b}{c\,d}$.

Square both members of (1), $\frac{a^2}{c^2} = \frac{b^2}{d^2}$.

By I. (Art. 163), $\frac{a\,b}{c\,d} = \frac{n\,a\,b}{n\,c\,d}$, or $\frac{a^2}{c^2} = \frac{n\,a\,b}{n\,c\,d}$.

Also I. (Art. 163), $\frac{b^2}{d^2} = \frac{p\,b^2}{p\,d^2}$, or $\frac{a^2}{c^2} = \frac{p\,b^2}{p\,d^2}$.

Also I. (Art. 163), $\frac{a^2}{c^2} = \frac{m\,a^2}{m\,c^2}$.

Hence, $\frac{m\,a^2}{m\,c^2} = \frac{p\,b^2}{p\,d^2} = \frac{n\,a\,b}{n\,c\,d} = \frac{b^2}{d^2} = \frac{a^2}{c^2}$.

By V. (Art. 163), $\frac{m\,a^2 + p\,b^2 + n\,a\,b}{m\,c^2 + p\,d^2 + n\,c\,d} = \frac{b^2}{d^2}$.

Therefore XI., $m\,a^2 + p\,b^2 + n\,a\,b : m\,c^2 + p\,d^2 + n\,c\,d :: b^2 : d^2$.

EXAMPLE 3. Find x when $\sqrt[3]{m+x} + \sqrt[3]{m-x} : \sqrt[3]{m+x} - \sqrt[3]{m-x} :: n : 1$.

Process. By VIII., $2\sqrt[3]{m+x} : 2\sqrt[3]{m-x} :: n+1 : n-1$,

or I. (Art. 163), $\sqrt[3]{m+x} : \sqrt[3]{m-x} :: n+1 : n-1$,

By X., $m+x : m-x :: (n+1)^3 : (n-1)^3$.

By I., $(m+x)(n-1)^3 = (m-x)(n+1)^3$.

Simplify, transpose, and factor, $2\,n\,(n^2+3)\,x = 2\,m\,(3\,n^2+1)$.

Therefore, $x = \frac{m\,(3\,n^2+1)}{n\,(n^2+3)}$.

Exercise 143.

If $a\,d = b\,c$, prove that:

1. $d : b :: c : a$; $d : c :: b : a$; $b : a :: d : c$.

2. $b : d :: a : c$; $c : a :: d : b$; $c : d :: a : b$.

Find a mean proportional between:

3. 2 and 8; 3 and $1\frac{1}{3}$; $1\frac{1}{5}$ and $\frac{2}{3}$; 8 and 18; a^3b and ab^3.

4. $(a+b)^2$ and $(a-b)^2$; $360\,a^4$ and $250\,a^2b^2$.

Find a third proportional to:

5. $\frac{3}{4}$ and $\frac{5}{6}$; $\frac{4}{3}$ and $\frac{2}{5}$; .2 and .4; 2 and 3; $\frac{5}{3}$ and $\frac{5}{7}$.

6. $\frac{1}{9}$ and $\sqrt{\frac{1}{3}}$; $(a-b)^2$ and a^2-b^2; $\dfrac{x}{y}+\dfrac{y}{x}$ and $\dfrac{x}{y}$.

Find a fourth proportional to:

7. 2, 5, and 6; 4, $\frac{3}{2}$, and $\frac{4}{3}$; $\frac{2}{7}$, $\frac{3}{4}$, and $\frac{5}{6}$; a, $a^{\frac{1}{2}}$, and b.

8. a^3, ab, and $5\,a^2b$; $\dfrac{x+1}{x-1}$, $\dfrac{x^2+x+1}{x^2-x+1}$, and $\dfrac{x^3+1}{x^3-1}$.

If $a:b::c:d$, prove that:

9. $a+b:a::c+d:c$; $a-b:a::c-d:c$.

10. $ac:bd::c^2:d^2$; $ab:cd::a^2:c^2$.

11. $2a+3c:3a+2c::2b+3d:3b+2d$.

12. $3a-5b:3c-5d::5a+3b:5c+3d$.

13. $\frac{2}{3}a:\frac{4}{5}b::\frac{2}{3}c:\frac{4}{5}d$; $\dfrac{a+b}{c+d}=\dfrac{a-b}{c-d}=\dfrac{a}{c}=\dfrac{b}{d}$.

14. $3a+2b:3a-2b::3c+2d:3c-2d$.

15. $la+mb:pa+qb::lc+md:pc+qd$.

16. $a^3:b^3::c^3:d^3$; $a^2:c^2::a^2-b^2:c^2-d^2$.

17. $a^2+c^2:ab+cd::ab+cd:b^2+d^2$.

18. $\sqrt{a+b}:\sqrt{b}::\sqrt{c+d}:\sqrt{d}$; $\dfrac{a}{c}=\sqrt{\dfrac{a^2+b^2}{c^2+d^2}}$.

If b is a mean proportional between a and c, prove that:

19. $a^2+2b^2 : a :: b^2+2c^2 : c;\quad \frac{b-c}{b} : \frac{a+b}{a} :: \frac{a-b}{a} : \frac{b+c}{b}$

If $a : b :: c : d :: e : f$, prove that:

20. $a : b :: a+c+e : b+d+f;\ \frac{a+2c+3e}{b+2d+3f} = \frac{2a+3c+4e}{2b+3d+4f}.$

21. d is a third proportional to a and b, and c is a third proportional to b and a, find a and b in terms of d and c.

If $m+n : m-n :: x+y : x-y$, prove that:

22. $x^2+m^2 : x^2-m^2 :: y^2+n^2 : y^2-n^2.$

Solve the following proportions:

23. $\left(\frac{10\,a^{\frac{2}{3}}}{3\,b^{\frac{5}{4}}}\right)^2 : x :: \sqrt{\frac{5\,a\,\sqrt[3]{a^2}}{4\,\sqrt[5]{a^2\,b^9}}} : \frac{9\,b^{-3}}{\sqrt{5}};\ 3^{2x} : 3^{x^2} :: \frac{1}{27} : 1.$

24. $x^3-y^3 : (x-y)^3 :: 19 : 1$ and $x : 6 :: 4 : y.$

25. If $\frac{a}{x-y} = \frac{b}{y-z} = \frac{c}{z-x}$, show that $a+b+c=0.$

26. A and B engage in business with different sums. A gains $1500, B loses $500, after which A's money is to B's as 3 to 2; but had A lost $500 and B gained $1000, then A's money would have been to B's as 5 to 9. Find each man's investment.

27. Show that the geometrical mean is a mean proportional between the arithmetical and harmonical means between the two numbers a and b.

28. When a, b, c, are in harmonical progression, show that $a : c :: a-b : b-c$. Hence, of three consecutive terms of a harmonical series, *the first is to the third as the first minus the second is to the second minus the third.*

APPENDIX.

COMPUTATION OF LOGARITHMS.

Since the logarithms of all composite numbers are found by adding the logarithms of their factors (Art. 122), it is only necessary to compute the logarithms of prime numbers.

The following method for computing logarithms is the one that was used when our tables were first made, although it is not the most expeditious method now known.

Example 1. Find the logarithm of 5.

Since $\quad 10^0 = 1,$

and $\quad 10^1 = 10 \qquad (1)$

and as 5 lies between 1 and 10, its logarithm must lie between 0 and 1.

Extract the square root of (1), $\quad 10^{.5} = 3.162277+ \qquad (2)$

As 5 lies between 10 and 3.162277+ its logarithm lies between 1 and .5.

Multiply (2) and (1) together, $\quad 10^{1.5} = 31.62277+.$

Take the square root, $\quad 10^{.75} = 5.623413+ \qquad (3)$

5 lies between 3.162277+ and 5.623413+, and its logarithm must lie between .5 and .75.

Multiply (2) and (3) together, $\quad 10^{1.25} = 17.7827895914+.$

Take the square root, $\quad 10^{.625} = 4.216964+ \qquad (4)$

Since 5 lies between 5.623413+ and 4.216964+, its logarithm must lie between .75 and .625.

Multiply (3) and (4) together, take the square root of the result, and we have $10^{.6875} = 4.869674+$. Continuing the process to 22 operations, we have, $\quad 10^{.698970+} = 5.000000+.$

Therefore, $\quad \log 5.000000+ = .698970+.$

EXAMPLE 2. Find the logarithm of 2.

$$\log 2 = \log \tfrac{10}{5} = \log 10 - \log 5 = 1 - .698970 = .301030.$$

EXAMPLE 3. Find the logarithm of 11.

	$10^1 = 10$	(1)
	$10^3 = 1000$	(2)
Extract the square root of (2),	$10^{1.5} = 31.62277+$	(3)
Multiply (3) and (1) together,	$10^{2.5} = 316.2277+.$	
Take the square root,	$10^{1.25} = 17.78278+$	(4)
Multiply (4) and (1) together,	$10^{2.25} = 177.8278+.$	
Take the square root,	$10^{1.125} = 13.33521+$	(5)
Multiply (5) and (1) together,	$10^{2.125} = 133.3521+.$	
Take the square root,	$10^{1.0625} = 11.54782-$	(6)
Multiply (6) and (1) together,	$10^{2.0625} = 115.4782+.$	
Take the square root,	$10^{1.03125} = 10.74607+$	(7)
Multiply (7) and (6) together,	$10^{2.09375} = 124.09368+.$	
Take the square root,	$10^{1.046875} = 11.13973+$	(8)
Multiply (8) and (7) together,	$10^{2.078125} = 119.70845+.$	
Take the square root,	$10^{1.0390625} = 10.94113+.$	
Therefore,	$\log 10.94113+ = 1.0390625.$	

Continuing the process, the logarithm of 11 may be found with sufficient accuracy.

EXAMPLE 4. Find the logarithm of 3.

Take $10^0 = 1$ and $10^{.5} = 3.162277+$, and proceed as before to 14 operations, and we have $\log 3.0000+ = .47712+$.

A table of logarithms to four decimal places will serve for many practical purposes. In the tables most generally used by computers they are given to six places of decimals. Seven to ten place logarithms are necessary for more accurate astronomical and mathematical calculations.

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